

# Constraints on Anomalous Fluid in Arbitrary Dimensions

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**ABSTRACT:** Using the techniques developed in arxiv: 1203.3544 we compute the universal part of the equilibrium partition function characteristic of a theory with multiple abelian  $U(1)$  anomalies in arbitrary even spacetime dimensions. This contribution is closely linked to the universal anomaly induced transport coefficients in hydrodynamics which have been studied before using entropy techniques. Equilibrium partition function provides an alternate and a microscopically more transparent way to derive the constraints on these transport coefficients. We re-derive this way all the known results on these transport coefficients including their polynomial structure which has recently been conjectured to be linked to the anomaly polynomial of the theory. Further we link the local description of anomaly induced transport in terms of a Gibbs current to the more global description in terms of the partition function.

**KEYWORDS:** Hydrodynamics, Anomaly.

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## 1. Introduction

Anomalies are a fascinating set of phenomena exhibited by field theories and string theories. For the sake of clarity let us begin by distinguishing between three quite different phenomena bearing that name.

The first phenomenon is when a symmetry of a classical action fails to be a symmetry at the quantum level. One very common example of an anomaly of this kind is the breakdown of classical scale invariance of a system when we consider the full quantum theory. This breakdown results in *renormalization group flow*, i.e., a scale-dependence of physical quantities even in a classically scale-invariant theory. Often this classical symmetry cannot be restored without seriously modifying the content of the theory. Anomalies of this kind often serve as a cautionary tale to remind us that the symmetries of a classical action like scale invariance will often not survive quantisation.

The second set of phenomena are what are termed as gauge anomalies. A system is said to exhibit a gauge anomaly if a particular classical gauge redundancy of the system is no more a redundancy at a quantum level. Since such redundancies are often crucial in eliminating unphysical states in a theory, a gauge anomaly often signifies a serious mathematical inconsistency in the theory. Hence this second kind of anomalies serve as a consistency criteria whereby we discard any theory exhibiting gauge anomaly as most probably inconsistent.

The third set of phenomena which we would be mainly interested in this work is when a genuine symmetry of a quantum theory is no more a symmetry when the theory is placed in a non-trivial background where we turn on sources for various operators in the theory. This lack of symmetry is reflected in the fact that the path integral with these sources turned on is no more invariant under the original symmetry transformations. If the sources are non-trivial gauge/gravitational backgrounds (corresponding to the charge/energy-momentum operators in the theory) the path integral is no more gauge-invariant. In fact as is well known the gauge transformation of the path-integral is highly constrained and the possible transformations are classified by the Wess-Zumino descent relations<sup>1</sup>.

Note that unlike the previous two phenomena here we make no reference to any specific classical description or the process of quantisation and hence this kind of anomalies are well-defined even in theories with multiple classical descriptions (or theories with no known classical description). Unlike the first kind of anomalies the symmetry is simply recovered at the quantum level by turning off the sources. Unlike the gauge anomalies the third kind of anomalies do not lead to any inconsistency. In what follows when

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<sup>1</sup>The Wess-Zumino descent relations are dealt with in detail in various textbooks[1, 2, 3] and lecture notes [4, 5].

we speak of anomaly we will always have in mind this last kind of anomalies unless specified otherwise.

Anomalies have been studied in detail in the least few decades and their mathematical structure and phenomenological consequence for zero temperature/chemical potential situations are reasonably well-understood. However the anomaly related phenomena in finite temperature setups let alone in non-equilibrium states are still relatively poorly understood despite their obvious relevance to fields ranging from solid state physics to cosmology. It is becoming increasingly evident that there are universal transport processes which are linked to anomalies present in a system and that study of anomalies provide a non-perturbative way of classifying these transport processes say in solid-state physics[6].

While the presence of transport processes linked to anomalies had been noticed before in a diversity of systems ranging from free fermions<sup>2</sup> to holographic fluids<sup>3</sup> a main advance was made in [12]. In that work it was shown using very general entropy arguments that the  $U(1)^3$  anomaly coefficient in an arbitrary  $3 + 1d$  relativistic field theory is linked to a specific transport process in the corresponding hydrodynamics. This argument has since then been generalised to finite temperature corrections [13, 14] and  $U(1)^{n+1}$  anomalies in  $d = 2n$  space time dimensions [15, 14]. In particular the author of [14] identified a rich structure to the anomaly-induced transport processes by writing down an underlying Gibbs-current which captured these processes in a succinct way. Later in a microscopic context in ideal Weyl gases, the authors of [8] identified this structure as emerging from an adiabatic flow of chiral states convected in a specific way in a given fluid flow.

While these entropy arguments are reasonably straightforward they appear somewhat non-intuitive from a microscopic field theory viewpoint. It is especially important to have a more microscopic understanding of these transport processes if one wants to extend the study of anomalies far away from equilibrium where one cannot resort to such thermodynamic arguments. So it is crucial to first rephrase these arguments in a more field theory friendly terms so that one may have a better insight on how to move far away from equilibrium.

Precisely such a field-theory friendly reformulation in  $3 + 1d$  and  $1 + 1d$  was found recently in the references [16] and [17] respectively. Our main aim in this paper is to generalise their results to arbitrary even space time dimensions. So let us begin by repeating the basic physical idea behind this reformulation in the next few paragraphs.

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<sup>2</sup>It would be an impossible task to list all the references in the last few decades which have discovered (and rediscovered) such effects in free/weakly coupled theories in various disguises using a diversity of methods. See for example [7] for what is probably the earliest study in  $3 + 1d$ . See [8] for a recent generalisation to arbitrary dimensions.

<sup>3</sup>See for example [9, 10, 11] for some of the initial holographic results.

Given a particular field theory exhibiting certain anomalies, one begins by placing that field theory in a time-independent gauge/gravitational background at finite temperature/chemical potential. We take the gauge/gravitational background to be spatially slowly varying compared to all other scales in the theory. Using this one can imagine integrating out all the heavy modes<sup>4</sup> in the theory to generate an effective Euler-Heisenberg type effective action for the gauge/gravitational background fields at finite temperature/chemical potential.

In the next step one expands this effective action in a spatial derivative expansion and then imposes the constraint that its gauge transformation be that fixed by the anomaly. This constrains the terms that can appear in the derivative expansion of the Euler-Heisenberg type effective action. As is clear from the discussion above, this effective action and the corresponding partition function have a clear microscopic interpretation in terms of a field-theory path integral and hence is an appropriate object in terms of which one might try to reformulate the anomalous transport coefficients.

The third step is to link various terms that appear in the partition function to the transport coefficients in the hydrodynamic equations. The crucial idea in this link is the realisation that the path integral we described above is essentially dominated by a time-independent hydrodynamic state (or more precisely a hydrostatic state). This means in particular that the expectation value of energy/momentum/charge/entropy calculated via the partition function should match with the distribution of these quantities in the corresponding hydrostatic state.

These distributions in turn depend on a subset of transport coefficients in the hydrodynamic constitutive relations which determine the hydrostatic state. In this way various terms that appear in the equilibrium partition function are linked to/constrain the transport coefficients crucial to hydrostatics. Focusing on just the terms in the path-integral which leads to the failure of gauge invariance we can then identify the universal transport coefficients which are linked to the anomalies. This gives a re derivation of various entropy argument results in a path-integral language thus opening the possibility that an argument in a similar spirit with Schwinger-Keldysh path integral will give us insight into non-equilibrium anomaly-induced phenomena.

Our main aim in this paper is twofold - first is to carry through in arbitrary dimensions this program of equilibrium partition function thus generalising the results of [16, 17] and re deriving in a path-integral friendly language the results of [15, 14].

Our second aim is to clarify the relation between the Gibbs current studied in [14, 8] and the partition function of [16, 17]. Relating them requires some care on carefully

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<sup>4</sup>Time-independence at finite temperature and chemical potential essentially means we are doing a Euclidean field theory. Unlike the Lorentzian field theory (which often has light-hydrodynamic modes) the Euclidean field theory has very few light modes except probably the Goldstone modes arising out of spontaneous symmetry breaking. We thank Shiraz Minwalla for emphasising this point.

distinguishing the consistent from covariant charge, the final result however is intuitive : the negative logarithm of the equilibrium partition function (times temperature) is simply obtained by integrating the equilibrium Gibbs free energy density (viz. the zeroth component of the Gibbs free current) over a spatial hyper surface. This provides a direct and an intuitive link between the local description in terms of a Gibbs current vs. the global description in terms of the partition function.

The plan of the paper is following. We will begin by mainly reviewing known results in Section §2. First we review the formalism/results of [14] in subsection§§2.1 where entropy arguments were used to constrain the anomaly-induced transport processes a Gibbs-current was written down which captured those processes in a succinct way. This is followed by subsection§§2.2 where we briefly review the relevant details of the equilibrium partition function formalism for fluids as developed in [16]. A recap of the relevant results in (3+1) and (1+1) dimensions[16, 17] and a comparison with results in this paper are relegated to appendix A.

Section §3 is devoted to the derivation of transport coefficients for  $2n$  dimensional anomalous fluid using the partition function method. The next section§4 contains construction of entropy current for the fluid and the constraints on it coming from partition function. This mirrors similar discussions in [16, 17]. We then compare these results to the results of [14] presented before in subsection§§2.1 and find a perfect agreement.

Prodged by this agreement, we proceed in next section§5 to a deeper analysis of the relation between the two formalisms. We prove an intuitive relation whereby the partition function could be directly derived from the Gibbs current of [14] by a simple integration (after one carefully shifts from the covariant to the consistent charge).

This is followed by section§6 where we generalise all our results for multiple  $U(1)$  charges. We perform a *CPT* invariance analysis of the fluid in section §7 and this imposes constraints on the fluid partition function. We end with conclusion and discussions in section§8.

Various technical details have been pushed to the appendices for the convenience of the reader. After the appendix A on comparison with previous partition function results in (3+1) and (1+1) dimensions, we have placed an appendix B detailing various specifics about the hydrostatic configuration considered in [16]. We then have an appendix C where we present the variational formulae to obtain currents from the partition function in the language of differential forms. This is followed by an appendix D on notations and conventions (especially the conventions of wedge product etc.).

## 2. Preliminaries

In this section we begin by reviewing and generalising various results from [14] where

constraints on anomaly-induced transport in arbitrary dimensions were derived using adiabaticity (i.e., the statement that there is no entropy production associated with these transport processes). Many of the zero temperature results here were also independently derived by the authors of [15].

We will then review the construction of equilibrium partition function (free energy) for fluid in the rest of the section. The technique has been well explained in [16] and familiar readers can skip this part.

## 2.1 Adiabaticity and Anomaly induced transport

Hydrodynamics is a low energy (or long wavelength) description of a quantum field theory around its thermodynamic equilibrium. Since the fluctuations are of low energy, we can express physical data in terms of derivative expansions of fluid variables (fluid velocity  $u(x)$ , temperature  $T(x)$  and chemical potential  $\mu(x)$ ) around their equilibrium value.

The dynamics of the fluid is described by some conservation equations. For example, the conservation equations of the fluid stress-tensor or the fluid charge current. These are known as constitutive equations. The stress tensor and charged current of fluid can be expressed in terms of fluid variables and their derivatives. At any derivative order, a generic form of the stress tensor and charged current can be written demanding symmetry and thermodynamics of the underlying field theory. These generic expressions are known as constitutive relations. As it turns out, validity of 2nd law of thermodynamics further constraints the form of these constitutive relations.

The author of [14] assumed the following form for the constitutive relations describing energy, charge and entropy transport in a fluid

$$\begin{aligned} T^{\mu\nu} &\equiv \varepsilon u^\mu u^\nu + p P^{\mu\nu} + q_{anom}^\mu u^\nu + u^\mu q_{anom}^\nu + T_{diss}^{\mu\nu} \\ J^\mu &\equiv q u^\mu + J_{anom}^\mu + J_{diss}^\mu \\ J_S^\mu &\equiv s u^\mu + J_{S,anom}^\mu + J_{S,diss}^\mu \end{aligned} \tag{2.1}$$

where  $u^\mu$  is the velocity of the fluid under consideration which obeys  $u^\mu u_\mu = -1$  when contracted using the space time metric  $g_{\mu\nu}$ . Further,  $P^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ , pressure of the fluid is  $p$  and  $\{\varepsilon, q, s\}$  are the energy, charge and the entropy densities respectively. We have denoted by  $\{q_{anom}^\mu, J_{anom}^\mu, J_{S,anom}^\mu\}$  the anomalous heat/charge/entropy currents and by  $\{T_{diss}^{\mu\nu}, J_{diss}^\mu, J_{S,diss}^\mu\}$  the dissipative currents.

### 2.1.1 Equation for adiabaticity

A convenient way to describe adiabatic transport process is via a **covariant** anomalous Gibbs current  $(\mathcal{G}_{anom}^{Cov})^\mu$ .

The adjective **covariant** refers to the fact that the Gibbs free energy and the corresponding partition function are computed by turning on chemical potential for

the **covariant** charge. This is to be contrasted with the **consistent** partition function and the corresponding **consistent** anomalous Gibbs current  $(\mathcal{G}_{anom}^{Consistent})^\mu$ .

Since this distinction is crucial let us elaborate this in the next few paragraphs - it is a fundamental result due to Noether that the continuous symmetries of a theory are closely linked to the conserved currents in that theory. Hence when the path integral fails to have a symmetry in the presence of background sources, there are two main consequences - first of all it directly leads to a modification of the corresponding charge conservation and a failure of Noether theorem. The second consequence is that various correlators obtained by varying the path integral are not gauge-covariant and a more general modifications of Ward identities occur.

A simple example is the expectation value of the current obtained by varying the path integral with respect to a gauge field (often termed the **consistent** current ) as,

$$J_{Consistent}^\mu \equiv \frac{\partial S}{\partial \mathcal{A}_\mu}.$$

The consistent current is not covariant under gauge transformation.

As has been explained in great detail in [18] thus there exists another current in anomalous theories: the covariant current. The covariant current  $J_{Cov}^\mu$  is a current shifted with respect to the consistent current by an amount  $J_c^\mu$ . The shift is such that its gauge transformation is anomalous and it exactly cancels the gauge non invariant part of the consistent current. Thus, the covariant current is covariant under the gauge transformation, as suggested by its name.

The covariant Gibbs current describes the transport of Gibbs free energy when a chemical potential is turned on for the covariant charge. We will take a Hodge-dual of this covariant Gibbs current to get a  $d - 1$  form in d-space time dimensions. Let us denote this Hodge-dual by  $\bar{\mathcal{G}}_{anom}^{Cov}$ . The anomalous parts of charge/entropy/energy currents can be derived from this Gibbs current via thermodynamics

$$\begin{aligned} \bar{J}_{anom}^{Cov} &= -\frac{\partial \bar{\mathcal{G}}_{anom}}{\partial \mu} \\ \bar{J}_{S,anom}^{Cov} &= -\frac{\partial \bar{\mathcal{G}}_{anom}}{\partial T} \\ \bar{q}_{anom}^{Cov} &= \bar{\mathcal{G}}_{anom} + T \bar{J}_{S,anom} + \mu \bar{J}_{anom} \end{aligned} \tag{2.2}$$

Then according to [14] the condition for adiabaticity is

$$d\bar{q}_{anom}^{Cov} + \mathbf{a} \wedge \bar{q}_{anom}^{Cov} - \mathcal{E} \wedge \bar{J}_{anom}^{Cov} = T d\bar{J}_{S,anom}^{Cov} + \mu d\bar{J}_{anom}^{Cov} - \mu \bar{\mathfrak{A}}^{Cov} \tag{2.3}$$

where  $\mathbf{a}, \mathcal{E}$  are the acceleration 1-form and the rest-frame electric field 1-form respectively defined via

$$\mathbf{a} \equiv (u \cdot \nabla) u_\mu dx^\mu, \quad \mathcal{E} \equiv u^\nu \mathcal{F}_{\mu\nu} dx^\mu$$



Further the rest frame magnetic field/vorticity 2-forms are defined by subtracting out the electric part from the gauge field strength and the acceleration part from the exterior derivative of velocity, viz.,

$$\mathcal{B} \equiv \mathcal{F} - u \wedge \mathcal{E} \ , \quad 2\omega \equiv du + u \wedge \mathfrak{a}$$

The symbol  $\bar{\mathfrak{A}}^{Cov}$  is the d-form which is the Hodge dual of the rate at which the **covariant** charge is created due to anomaly,i.e.,

$$d\bar{J}^{Cov} = \bar{\mathfrak{A}}^{Cov}$$

where  $\bar{J}^{Cov}$  is the entire covariant charge current including both the anomalous and the non-anomalous pieces. For simplicity we have restricted our attention to a single U(1) global symmetry which becomes anomalous on a non-trivial background.

In terms of the Gibbs current , we can write the adiabaticity condition (2.3) as,

$$d\bar{\mathcal{G}}_{anom}^{Cov} + \mathfrak{a} \wedge \bar{\mathcal{G}}_{anom}^{Cov} + \mu \bar{\mathfrak{A}}^{Cov} = (dT + \mathfrak{a}T) \wedge \frac{\partial \bar{\mathcal{G}}_{anom}^{Cov}}{\partial T} + (d\mu + \mathfrak{a}\mu - \mathcal{E}) \wedge \frac{\partial \bar{\mathcal{G}}_{anom}^{Cov}}{\partial \mu} \quad (2.4)$$

### 2.1.2 Construction of the polynomial $\mathfrak{F}_{anom}^\omega$

The main insight of [14] is that in d-space time dimensions the solutions of this equation are most conveniently phrased in terms of a single homogeneous polynomial of degree  $n + 1$  in temperature  $T$  and chemical potential  $\mu$ .

Following the notation employed in [8] we will denote this polynomial as  $\mathfrak{F}_{anom}^\omega[T, \mu]$ . As was realised in [8], this polynomial is often closely related to the anomaly polynomial of the system<sup>5</sup>. More precisely, for a variety of systems we have a remarkable relation between  $\mathfrak{F}_{anom}^\omega[T, \mu]$  and the anomaly polynomial  $\mathcal{P}_{anom}[\mathcal{F}, \mathfrak{R}]$

$$\mathfrak{F}_{anom}^\omega[T, \mu] = \mathcal{P}_{anom}[\mathcal{F} \mapsto \mu, p_1(\mathfrak{R}) \mapsto -T^2, p_{k>1}(\mathfrak{R}) \mapsto 0] \quad (2.5)$$

Let us be more specific : on a  $(2n - 1) + 1$  dimensional space time consider a theory with

$$\mathfrak{F}_{anom}^\omega[T, \mu] = \mathcal{C}_{anom} \mu^{n+1} + \sum_{m=0}^n C_m T^{m+1} \mu^{n-m} \quad (2.6)$$

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<sup>5</sup>We remind the reader that the anomalies of a theory living in  $d = 2n$  spacetime dimensions is succinctly captured by a  $2n + 2$  form living in *two dimensions higher*. This  $2n + 2$  form called the anomaly polynomial (since it is a polynomial in external/background field strengths  $\mathcal{F}$  and  $\mathfrak{R}$ ) is related to the variation of the effective action  $\delta W$  via *the descent relations*

$$\mathcal{P}_{anom} = d\Gamma_{CS} \ , \quad \delta\Gamma_{CS} = d\delta W$$

We will refer the reader to various textbooks[1, 2, 3] and lecture notes [4, 5] for a more detailed exposition.

Assuming that the theory obeys the replacement rule (2.5) such a  $\mathfrak{F}_{anom}^\omega[T, \mu]$  can be obtained from an anomaly polynomial<sup>6</sup>

$$\mathcal{P}_{anom} = \mathcal{C}_{anom} \mathcal{F}^{n+1} + \sum_{m=0}^n C_m [-p_1(\mathfrak{R})]^{\frac{m+1}{2}} \mathcal{F}^{n-m} + \dots \quad (2.7)$$

where we have presented the terms which do not involve the higher Pontryagin forms. Restricting our attention only to the  $U(1)^{n+1}$  anomaly (and ignoring the mixed/pure gravitational anomalies ) we can write

$$\begin{aligned} d\bar{J}_{Consistent} &= \mathcal{C}_{anom} \mathcal{F}^n \\ d\bar{J}_{Cov} &= (n+1) \mathcal{C}_{anom} \mathcal{F}^n \end{aligned} \quad (2.8)$$

and their difference is given by

$$\bar{J}_{Cov} = \bar{J}_{Consistent} + n \mathcal{C}_{anom} \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} \quad (2.9)$$

The solution of (2.4) corresponding to the homogeneous polynomial (2.6) is given by

$$\begin{aligned} \bar{\mathcal{G}}_{anom}^{Cov} &= C_0 T \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} + \sum_{m=1}^n \left[ \mathcal{C}_{anom} \binom{n+1}{m+1} \mu^{m+1} \right. \\ &\quad \left. + \sum_{k=0}^m C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k} \right] (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \end{aligned} \quad (2.10)$$

Here  $\hat{\mathcal{A}}$  is the  $U(1)$  gauge-potential 1-form in some gauge with  $\mathcal{F} \equiv d\hat{\mathcal{A}}$  being its field-strength 2-form. Further,  $\mathcal{B}, \omega$  are the rest frame magnetic field/vorticity 2-forms and  $T, \mu$  are the local temperature and chemical potential respectively. They obey

$$(d\mathcal{B}) \wedge u = -(2\omega) \wedge \mathcal{E} \wedge u, \quad d(2\omega) \wedge u = (2\omega) \wedge \mathfrak{a} \wedge u \quad (2.11)$$

Using these equations it is a straightforward exercise to check that (2.10) furnishes a solution to (2.4).

We will make a few remarks before we proceed to derive charge/entropy/energy currents from this Gibbs current. Note that if one insists that the Gibbs current be gauge-invariant then we are forced to put  $C_0 = 0$  - in the solution presented in [14] this condition was implicitly assumed and the  $C_0$  term was absent. The authors of [16] later relaxed this assumption insisting gauge-invariance only for the covariant charge/energy

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<sup>6</sup>Since all relativistic theories only have integer powers of Pontryagin forms the constants  $C_m$  should vanish whenever  $m$  is even. As we shall see later that another way to arrive at the same conclusion is to impose CPT invariance.

currents. Since we would be interested in comparison with the results derived in [16] it is useful to retain the  $C_0$  term.

Now we use thermodynamics to obtain the charge current as

$$\begin{aligned} \bar{J}_{anom}^{Cov} &= - \sum_{m=1}^n \left[ (m+1) \mathcal{C}_{anom} \binom{n+1}{m+1} \mu^m \right. \\ &\quad \left. + \sum_{k=0}^m (m-k) C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k-1} \right] (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \end{aligned} \quad (2.12)$$

and the entropy current is given by

$$\begin{aligned} \bar{J}_{S,anom}^{Cov} &= -C_0 \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} \\ &\quad - \sum_{m=1}^n \sum_{k=0}^m (k+1) C_k \binom{n-k}{m-k} T^k \mu^{m-k} (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \end{aligned} \quad (2.13)$$

The energy current is given by

$$\begin{aligned} \bar{q}_{anom}^{Cov} &= - \sum_{m=1}^n m \left[ \mathcal{C}_{anom} \binom{n+1}{m+1} \mu^{m+1} \right. \\ &\quad \left. + \sum_{k=1}^m C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k} \right] (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \end{aligned} \quad (2.14)$$

These currents satisfy an interesting Reciprocity type relationship noticed in [14]

$$\frac{\delta \bar{q}_{anom}^{Cov}}{\delta \mathcal{B}} = \frac{\delta \bar{J}_{anom}^{Cov}}{\delta (2\omega)} \quad (2.15)$$

While this is a solution in a generic frame one can specialise to the Landau frame (where the velocity is defined via the energy current) by a frame transformation

$$\begin{aligned} u^\mu &\mapsto u^\mu - \frac{q_{anom}^\mu}{\epsilon + p}, \\ J_{anom}^\mu &\mapsto J_{anom}^\mu - q \frac{q_{anom}^\mu}{\epsilon + p}, \\ J_{S,anom}^\mu &\mapsto J_{S,anom}^\mu - s \frac{q_{anom}^\mu}{\epsilon + p}, \\ q_{anom}^\mu &\mapsto 0 \end{aligned} \quad (2.16)$$

to get

$$\begin{aligned} \bar{J}_{anom}^{Cov, Landau} &= \sum_{m=1}^n \xi_m (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \\ \bar{J}_{S,anom}^{Cov, Landau} &= \sum_{m=1}^n \xi_m^{(s)} (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u + \zeta \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} \end{aligned} \quad (2.17)$$

where

$$\begin{aligned}
\xi_m &\equiv \left[ m \frac{q\mu}{\epsilon + p} - (m+1) \right] \mathcal{C}_{anom} \binom{n+1}{m+1} \mu^m \\
&\quad + \sum_{k=0}^m \left[ m \frac{q\mu}{\epsilon + p} - (m-k) \right] C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k-1} \\
\xi_m^{(s)} &\equiv \left[ m \frac{sT}{\epsilon + p} \right] \mathcal{C}_{anom} \binom{n+1}{m+1} T^{-1} \mu^{m+1} \\
&\quad + \sum_{k=0}^m \left[ m \frac{sT}{\epsilon + p} - (k+1) \right] C_k \binom{n-k}{m-k} T^k \mu^{m-k} \\
\zeta &= -C_0
\end{aligned} \tag{2.18}$$

Often in the literature the entropy current is quoted in the form

$$\bar{J}_{S,anom}^{Cov,Landau} = -\frac{\mu}{T} \bar{J}_{anom}^{Cov,Landau} + \sum_{m=1}^n \chi_m (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u + \zeta \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} \tag{2.19}$$

where

$$\begin{aligned}
\zeta &= -C_0 \\
\chi_m &\equiv \xi_m^{(s)} + \frac{\mu}{T} \xi_m \\
&= -\mathcal{C}_{anom} \binom{n+1}{m+1} T^{-1} \mu^{m+1} - \sum_{k=0}^m C_k \binom{n-k}{m-k} T^k \mu^{m-k}
\end{aligned} \tag{2.20}$$

where we have used the thermodynamic relation  $sT + q\mu = \epsilon + p$ . By looking at (2.10) we recognise these to be the coefficients occurring in the anomalous Gibbs current :

$$\bar{\mathcal{G}}_{anom}^{Cov} = -T \left[ \sum_{m=1}^n \chi_m (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u + \zeta \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} \right] \tag{2.21}$$

In fact this is to be expected from basic thermodynamic considerations : the above equation is a direct consequence of the relation  $G = -T(S + \frac{\mu}{T}Q - \frac{U}{T})$  and the fact that energy current receives no anomalous contributions in the Landau frame.

This ends our review of the main results of [14] adopted to our purposes. Our aim in the rest of the paper would be to derive all these results purely from a partition function analysis.

## 2.2 Equilibrium Partition Function

In this subsection we review (and extension) an alternative approach to constrain the constitutive relations, namely by demanding the existence of an equilibrium partition function (or free energy) for the fluid as described in [16, 17] <sup>7</sup>.

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<sup>7</sup>For similar discussions, see for example [19, 20].

Let us keep the fluid in a special background such that the background metric has a time like killing vector and the background gauge field is time independent. Any such metric can be put into the following Kaluza-Klein form

$$\begin{aligned} ds^2 &= -e^{2\sigma}(dt + a_i dx^i)^2 + g_{ij} dx^i dx^j, \\ \hat{\mathcal{A}} &= \mathcal{A}_0 dt + \mathcal{A}_i dx^i \end{aligned} \quad (2.22)$$

here  $i, j \in (1, 2 \dots 2n-1)$  are the spatial indices. We will often use the notation  $\gamma \equiv e^{-\sigma}$  for brevity. This background has a time-like killing vector  $\partial_t$  and let  $u_k^\mu = (e^{-\sigma}, 0, 0, \dots)$  be the unit normalized vector in the killing direction so that

$$u_k^\mu \partial_\mu = \gamma \partial_t \quad \text{and} \quad u_k = -\gamma^{-1}(dt + a)$$

In the corresponding Euclidean field theory description of equilibrium, the imaginary time direction would be compactified into a thermal circle with the size of circle being the inverse temperature of the underlying field theory. In the  $2n-1$  dimensional compactified geometry, the original  $2n$  background field breaks as follow

- metric( $g_{\mu\nu}$ ) : scalar( $\sigma$ ), KK gauge field( $a_i$ ), lower dimensional metric( $g_{ij}$ ).
- gauge field( $\hat{\mathcal{A}}_\mu$ ) : scalar( $\mathcal{A}_0$ ), gauge field( $\mathcal{A}_i$ )

Under this KK type reduction the  $2n$  dimensional diffeomorphisms breaks up into  $2n-1$  dimensional diffeomorphisms and KK gauge transformations. The components of  $2n$  dimensional tensors which are KK-gauge invariant in  $2n-1$  dimensions are those with lower time(killing direction) and upper space indices. Given a 1-form  $J$  we will split it in terms of KK-invariant components as

$$J = J_0(dt + a_i dx^i) + g_{ij} J^i dx^j$$

Other KK non-invariant components of  $J$  are given by

$$\begin{aligned} J^0 &= -[\gamma^2 J_0 + a_i J^i] \\ J_i &= g_{ij} J^j + a_i J_0 \end{aligned} \quad (2.23)$$

To take care of KK gauge invariance we will identify the lower dimensional U(1) gauge field (denoted by non script letters) as follows

$$\begin{aligned} A_0 &= \mathcal{A}_0 + \mu_0, \quad A^i = \mathcal{A}^i \\ \Rightarrow A_i &= \mathcal{A}_i - \mathcal{A}_0 a_i \quad \text{and} \\ F_{ij} &= \partial_i A_j - \partial_j A_i = \mathcal{F}_{ij} - A_0 f_{ij} - (\partial_i A_0 a_j - \partial_j A_0 a_i). \end{aligned} \quad (2.24)$$

where  $f_{ij} \equiv \partial_i a_j - \partial_j a_i$  and  $\mu_0$  is a convenient constant shift in  $\mathcal{A}_0$  which we will define shortly. We can hence write

$$\hat{\mathcal{A}} = \mathcal{A}_0 dt + \mathcal{A} = A_0(dt + a_i dx^i) + A_i dx^i - \mu_0 dt$$

We are now working in a general gauge - often it is useful to work in a specific class of gauges : one class of gauges we will work on is obtained from this generic gauge by performing a gauge transformation to remove the  $\mu_0 dt$  piece. We will call these class of gauges as the ‘zero  $\mu_0$ ’ gauges. In these gauges the new gauge field is given in terms of the old gauge field via

$$\hat{\mathcal{A}}_{\mu_0=0} \equiv \hat{\mathcal{A}} + \mu_0 dt$$

We will quote all our consistent currents in this gauge. The field strength 2-form can then be written as

$$\mathcal{F} \equiv d\hat{\mathcal{A}} = dA + A_0 da + dA_0 \wedge (dt + a)$$

We will now focus our attention on the **consistent** equilibrium partition function which is the Euclidean path-integral computed on space adjoined with a thermal circle of length  $1/T_0$ . We will further turn on a chemical potential  $\mu$  - since there are various different notions of charge in anomalous theories placed in gauge backgrounds we need to carefully define which of these notions we use to define the partition function<sup>8</sup>. While in the previous subsection we used the chemical potential for a **covariant** charge and the corresponding **covariant** Gibbs free-energy following [14] , in this subsection we will follow [16] in using a chemical potential for the consistent charge to define the partition function. This distinction has to be kept in mind while making a comparison between the two formalisms as we will elaborate later in section§5.

The consistent partition function  $Z_{Consistent}$  that we write down will be the most general one consistent with  $2n-1$  dimensional diffeomorphisms, KK gauge invariance and the  $U(1)$  gauge invariance up to anomaly. It is a scalar  $S$  constructed out of various background quantities and their derivatives. The most generic form of the partition function is

$$W = \ln Z_{Consistent} = \int d^{2n-1}x \sqrt{g_{2n-1}} S(\sigma, A_0, a_i, A_i, g_{ij}). \quad (2.25)$$

Given this partition function, we compute various components of the stress tensor and charged current from it. The KK gauge invariant components of the stress tensor  $T_{\mu\nu}$  and charge current  $J_\mu$  can then be obtained from the partition function as follows [16],

$$\begin{aligned} T_{00} &= -\frac{T_0 e^{2\sigma}}{\sqrt{-g_{2n}}} \frac{\delta W}{\delta \sigma}, \quad J_0^{Consistent} = -\frac{e^{2\sigma} T_0}{\sqrt{-g_{2n}}} \frac{\delta W}{\delta A_0}, \\ T_0^i &= \frac{T_0}{\sqrt{-g_{2n}}} \left( \frac{\delta W}{\delta a_i} - A_0 \frac{\delta W}{\delta A_i} \right), \quad J_{Consistent}^i = \frac{T_0}{\sqrt{-g_{2n}}} \frac{\delta W}{\delta A_i}, \\ T^{ij} &= -\frac{2T_0}{\sqrt{-g_{2n}}} g^{il} g^{jm} \frac{\delta W}{\delta g^{lm}}. \end{aligned} \quad (2.26)$$

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<sup>8</sup>See, for example, section§3 of [21] for a discussion of some of the subtleties.

here  $\{\sigma, a_i, g_{ij}, A_0, A_i\}$  are chosen independent sources, so the partial derivative w.r.t any of them in the above equations means that others are kept constant. We will sometimes use the above equation written in terms of differential forms - we will refer the reader to appendix C for the differential-form version of the above equations.

Next we parameterize the most generic equilibrium solution and constitutive relations for the fluid as,

$$\begin{aligned} u(x) &= u_0(x) + u_1(x), \quad T(x) = T_0(x) + T_1(x), \quad \mu(x) = \mu_0(x) + \mu_1(x), \\ T_{\mu\nu} &= (\epsilon + p)u_\mu u_\nu + pg_{\mu\nu} + \pi_{\mu\nu}, \quad J^\mu = qu^\mu + j_{diss}^\mu, \end{aligned} \quad (2.27)$$

where,  $u_1, T_1, \mu_1, \pi_{\mu\nu}, j_{diss}^\mu$  are various derivatives of the background quantities. Note that we will work in Landau frame throughout.

These corrections are found by comparing the fluid stress tensor  $T_{\mu\nu}$  and current  $J_\mu$  in Eqn.(2.27) with  $T_{\mu\nu}$  and  $J_\mu$  in Eqn.(2.26) as obtained from the partition function. This exercise then constrains various non-dissipative coefficients that appear in the constitutive relations in Eqn.(2.27).

This then ends our short review of the formalism developed in [16]. In the next section we will apply this formalism to a theory with  $U(1)^{n+1}$  anomaly in  $d = 2n$  space time dimensions.

### 3. Anomalous partition function in arbitrary dimensions

Let us consider then a fluid in a  $2n$  dimensional space time. The fluid is charged under a single  $U(1)$  abelian gauge field  $\mathcal{A}_\mu$ . We will generalise to multiple abelian gauge fields later in section §6 and leave the non-abelian case for future study. We will continue to use the notation in the subsection §§2.1.

The consistent/covariant anomaly are then given by Eqn.(2.8) which can be written in components as

$$\begin{aligned} \nabla_\mu J_{Consistent}^\mu &= \mathcal{C}_{anom} \varepsilon^{\mu_1 \nu_1 \dots \mu_n \nu_n} \partial_{\mu_1} \hat{\mathcal{A}}_{\nu_1} \dots \partial_{\mu_n} \hat{\mathcal{A}}_{\nu_n} \\ &= \frac{\mathcal{C}_{anom}}{2^n} \varepsilon^{\mu_1 \nu_1 \dots \mu_n \nu_n} \mathcal{F}_{\mu_1 \nu_1} \dots \mathcal{F}_{\mu_n \nu_n}. \\ \nabla_\mu J_{Cov}^\mu &= (n+1) \mathcal{C}_{anom} \varepsilon^{\mu_1 \nu_1 \dots \mu_n \nu_n} \partial_{\mu_1} \hat{\mathcal{A}}_{\nu_1} \dots \partial_{\mu_n} \hat{\mathcal{A}}_{\nu_n} \\ &= (n+1) \frac{\mathcal{C}_{anom}}{2^n} \varepsilon^{\mu_1 \nu_1 \dots \mu_n \nu_n} \mathcal{F}_{\mu_1 \nu_1} \dots \mathcal{F}_{\mu_n \nu_n}. \end{aligned} \quad (3.1)$$

and Eqn.(2.9) becomes

$$J_{Cov}^\mu = J_{Consistent}^\mu + J_{(c)}^\mu. \quad (3.2)$$

where

$$\begin{aligned} J_{(c)}^\lambda &= n \mathcal{C}_{anom} \varepsilon^{\lambda \alpha \mu_1 \nu_1 \dots \mu_{n-1} \nu_{n-1}} \hat{\mathcal{A}}_\alpha \partial_{\mu_1} \hat{\mathcal{A}}_{\nu_1} \dots \partial_{\mu_{n-1}} \hat{\mathcal{A}}_{\nu_{n-1}} \\ &= n \frac{\mathcal{C}_{anom}}{2^{n-1}} \varepsilon^{\lambda \alpha \mu_1 \nu_1 \dots \mu_{n-1} \nu_{n-1}} \hat{\mathcal{A}}_\alpha \mathcal{F}_{\mu_1 \nu_1} \dots \mathcal{F}_{\mu_{n-1} \nu_{n-1}}. \end{aligned} \quad (3.3)$$

The energy-momentum equation becomes

$$\nabla_\mu T_\nu^\mu = F_{\nu\mu} J_{Cov}^\mu, \quad (3.4)$$

where  $J_{Cov}^\mu$  is the covariant current. This has been explicitly shown in [16]<sup>9</sup>.

### 3.1 Constraining the partition function

We want to write the equilibrium free energy functional for the fluid. For this purpose, let us keep the in the following  $2n$ -dimensional time independent background,

$$ds^2 = -e^{2\sigma}(dt + a_i dx^i)^2 + g_{ij} dx^i dx^j, \quad \mathcal{A} = (A_0, \mathcal{A}_i). \quad (3.5)$$

Now, we write the  $(2n - 1)$  dimensional equilibrium free energy that reproduces the same anomaly as given in (6.2). The most generic form for the anomalous part of the partition function is ,

$$W_{anom} = \frac{1}{T_0} \int d^{2n-1}x \sqrt{g_{2n-1}} \left\{ \sum_{m=1}^n \alpha_{m-1}(A_0, T_0) [\epsilon A(da)^{m-1}(dA)^{n-m}] \right. \\ \left. + \alpha_n(T_0) [\epsilon a(da)^{n-1}] \right\}. \quad (3.6)$$

where,  $\epsilon^{ijk\dots}$  is the  $(2n - 1)$  dimensional tensor density defined via

$$\epsilon^{i_1 i_2 \dots i_{d-1}} = e^{-\sigma} \epsilon^{0 i_1 i_2 \dots i_{d-1}}$$

The indices  $(i, j)$  run over  $(2n - 1)$  values. We have used the following notation for the sake of brevity

$$\begin{aligned} & [\epsilon A(da)^{m-1}(dA)^{n-m}] \\ & \equiv \epsilon^{ij_1 k_1 \dots j_{m-1} k_{m-1} p_1 q_1 \dots p_{n-m} q_{n-m}} A_i \partial_{j_1} a_{k_1} \dots \partial_{j_{m-1}} a_{k_{m-1}} \partial_{p_1} A_{q_1} \dots \partial_{p_{n-m}} A_{q_{n-m}} \\ & [\epsilon (da)^{m-1}(dA)^{n-m}]^i \\ & \equiv \epsilon^{ij_1 k_1 \dots j_{m-1} k_{m-1} p_1 q_1 \dots p_{n-m} q_{n-m}} \partial_{j_1} a_{k_1} \dots \partial_{j_{m-1}} a_{k_{m-1}} \partial_{p_1} A_{q_1} \dots \partial_{p_{n-m}} A_{q_{n-m}} \end{aligned} \quad (3.7)$$

The invariance under diffeomorphism implies that  $\alpha_n$  is a constant in space .For  $m < n$  however  $\alpha_m$  can have  $A_0$  dependence, as the gauge symmetry is anomalous, but they are independent of  $\sigma$ , due to diffiomorphism invariance.

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<sup>9</sup>One required identity is,

$$\hat{\mathcal{A}}_\alpha \varepsilon^{\mu_1 \nu_1 \dots \mu_n \nu_n} \mathcal{F}_{\mu_1 \nu_1} \dots \mathcal{F}_{\mu_n \nu_n} = 2n \hat{\mathcal{A}}_{\mu_1} \varepsilon^{\mu_1 \nu_1 \mu_2 \nu_2 \dots \mu_n \nu_n} \mathcal{F}_{\alpha \nu_1} \mathcal{F}_{\mu_2 \nu_2} \dots \mathcal{F}_{\mu_n \nu_n}$$

for arbitrary  $2n$ -dimensions



The consistent current computed from this partition function is,

$$\begin{aligned}
(J_{anom})_0^{Consistent} &= -e^\sigma \sum_{m=1}^n \frac{\partial \alpha_{m-1}}{\partial A_0} [\epsilon A (da)^{m-1} (dA)^{n-m}] \\
(J_{anom})_{Consistent}^i &= e^{-\sigma} \left\{ \sum_{m=1}^n (n-m+1) \alpha_{m-1} [\epsilon (da)^{m-1} (dA)^{n-m}]^i \right. \\
&\quad \left. - \sum_{m=1}^{n-1} (n-m) \frac{\partial \alpha_{m-1}}{\partial A_0} [\epsilon A dA_0 (da)^{m-1} (dA)^{n-m-1}]^i \right\}
\end{aligned} \tag{3.8}$$

Next, we compute the covariant currents, following (3.2). The correction piece for the 0-component of the current is,

$$(J_{(c)})_0 = -n \mathcal{C}_{anom} e^\sigma \sum_{m=1}^n A_0^m \binom{n-1}{m-1} [\epsilon A (da)^{m-1} (dA)^{n-m}] \tag{3.9}$$

where, we have used the following identification for  $2n$  dimensional gauge field  $\mathcal{A}_\mu$  and  $(2n-1)$  dimensional gauge fields  $A_i, a_i$  and scalar  $A_0$ ,

$$\begin{aligned}
\mathcal{A}_i &= A_i + a_i A_0 \\
\mathcal{A}_0 &= A_0.
\end{aligned} \tag{3.10}$$

where we are working in a ‘zero  $\mu_0$ ’ gauge.

Thus, the 0-component of the covariant current is,

$$(J_{anom})_0^{Cov} = -e^\sigma \epsilon^{ijkl\dots} \sum_{m=1}^n \left[ \frac{\partial \alpha_{m-1}}{\partial A_0} + n \binom{n-1}{m-1} A_0^{m-1} \mathcal{C}_{anom} \right] [\epsilon A (da)^{m-1} (dA)^{n-m}]. \tag{3.11}$$

Every term in the above sum is gauge non-invariant. So the covariance of the covariant current demands that we chose the arbitrary functions  $\alpha_m$  appearing in the partition function (3.6) such that the current vanishes. Thus, we get,

$$\frac{\partial \alpha_{m-1}}{\partial A_0} + n \binom{n-1}{m-1} A_0^{m-1} \mathcal{C}_{anom} = 0. \tag{3.12}$$

The solution for the above equation is,

$$\begin{aligned}
\alpha_m &= -\mathcal{C}_{anom} \binom{n}{m+1} A_0^{m+1} + \tilde{C}_m T_0^{m+1}, \quad m = 0, \dots, n-1 \\
\alpha_n &= \tilde{C}_n T_0^{n+1}
\end{aligned} \tag{3.13}$$

Here,  $\tilde{C}_m$  are constants that can appear in the partition function.

Thus, at this point, a total of  $n+1$  coefficients can appear in the partition function. A further study of CPT invariance of the partition function will reduce this number. We will present that analysis later in details and here we just state the result. CPT forces all  $\tilde{C}_{2k} = 0$ . For even  $n$ , the number of constants are  $\frac{n}{2}$  where as for odd  $n$ , the number is  $(\frac{n+1}{2})$ .

### 3.2 Currents from the partition function

With these functions the  $i$ -component of the covariant current is,

$$\begin{aligned}
(J_{anom})_{Cov}^i &= e^{-\sigma} \sum_{m=1}^n \left[ A_0 \frac{\partial \alpha_{m-1}}{\partial A_0} + (n-m+1) \alpha_{m-1} \right] [\epsilon (da)^{m-1} (dA)^{n-m}]^i \\
&= e^{-\sigma} \sum_{m=1}^n \left[ - (n+1) \mathcal{C}_{anom} \binom{n}{m} T_0 A_0^m \right. \\
&\quad \left. + (n-m+1) T_0^m \tilde{C}_{m-1} \right] [\epsilon (da)^{m-1} (dA)^{n-m}]^i,
\end{aligned} \tag{3.14}$$

As expected, this current is  $U(1)$  gauge invariant. The different components of stress-tensor computed from the partition function are,

$$\begin{aligned}
T_{00}^{anom} &= 0, \quad T_{anom}^{ij} = 0 \\
(T_0^i)_{anom} &= e^{-\sigma} \sum_{m=1}^n (m \alpha_m - (n-m+1) A_0 \alpha_{m-1}) [\epsilon (da)^{m-1} (dA)^{n-m}]^i \\
&= e^{-\sigma} \sum_{m=1}^n \left[ m \tilde{C}_m T_0^{m+1} - (n+1-m) \tilde{C}_{m-1} T_0^m A_0 \right. \\
&\quad \left. + \binom{n+1}{m+1} \mathcal{C}_{anom} A_0^{m+1} \right] [\epsilon (da)^{m-1} (dA)^{n-m}]^i
\end{aligned} \tag{3.15}$$

### 3.3 Comparison with Hydrodynamics

Next, we find the equilibrium solution for the fluid variables. As usual, we keep the fluid in the time independent background (3.5). The equilibrium solutions for perfect charged fluid (with out any dissipation) are,

$$u^\mu \partial_\mu = e^{-\sigma} \partial_t, \quad T = T_0 e^{-\sigma}, \quad \mu = A_0 e^{-\sigma}. \tag{3.16}$$

The most generic constitutive relations for the fluid can be written as,

$$\begin{aligned}
T_{\mu\nu} &= (\epsilon + p) u_\mu u_\nu + p g_{\mu\nu} + \eta \sigma_{\mu\nu} + \zeta \Theta \mathcal{P}_{\mu\nu} \\
J_{Cov}^\mu &= q u^\mu + J_{even}^\mu + J_{odd}^\mu, \\
J_{even}^\mu &= \sigma (E^\mu - T \mathcal{P}^{\mu\alpha} \partial_\alpha \nu) + \alpha_1 E^\mu + \alpha_2 T \mathcal{P}^{\mu\alpha} \partial_\alpha \nu + \text{higher derivative terms} \\
J_{odd}^\mu &= \sum_{m=1}^n \xi_m \varepsilon^{\mu\nu \gamma_1 \delta_1 \dots \gamma_{m-1} \delta_{m-1} \alpha_1 \beta_1 \dots \alpha_{n-m} \beta_{n-m}} u_\nu (\partial_\gamma u_\delta)^{m-1} (\partial_\alpha \mathcal{A}_\beta)^{n-m} + \dots
\end{aligned} \tag{3.17}$$

Here,  $J_{even}^\mu$  is parity even part of the charge current and  $J_{odd}^\mu$  is parity odd charge current.  $\varepsilon^{\mu\nu\alpha\beta\gamma\delta\dots}$  is a  $2n$  dimensional tensor density whose  $(n-m)$  indices are contracted with  $\partial_\alpha \mathcal{A}_\beta$  and  $(m-1)$  indices are contracted with  $\partial_\gamma u_\delta$ .

We notice that the higher derivative part of the current gets contribution from both parity even and odd vectors. Parity even vectors can be at any derivative order but

parity odd vectors always appear at  $(n - 1)$  derivative order. Thus, for a generic value of  $n$  (other than  $n = 2$ ), the parity even and odd parts corrections to the current will always appear at different derivative orders. From now on, we will only concentrate on the parity odd sector. It is also straight forward to check that  $J_0^{odd} = 0$ .

Next, we look for the equilibrium solution for this fluid. Since, there exist no gauge invariant parity odd scalar, the temperature and chemical potential do not get any correction. Also, in  $2n$  dimensional theory, the parity odd vectors that we can write are always  $(n - 1)$  derivative terms. No other parity odd vector at any lower derivative order exists. Since the fluid velocity is always normalized to unity, we have,

$$\delta T = 0, \quad \delta \mu = 0, \quad \delta u_0 = -a_i \delta u^i. \quad (3.18)$$

where, the most generic correction to the fluid velocity is,

$$\delta u^i = \sum_{m=1}^n U_m(\sigma, A_0) [\epsilon (da)^{m-1} (dA)^{n-m}]^i. \quad (3.19)$$

Here,  $U_m(\sigma, A_0)$  are arbitrary coefficients and factors of  $e^\sigma$  is introduced for later convenience. Similarly, we can parameterize the  $i$ -component of the parity-odd current as,

$$J_{odd}^i = \sum_{m=1}^n J_m(\sigma, A_0) [\epsilon (da)^{m-1} (dA)^{n-m}]^i. \quad (3.20)$$

The coefficients  $J_m(\sigma, A_0)$  are related to the transport coefficients  $\xi_m$  via

$$J_m = \sum_{k=1}^m \binom{n-k}{m-k} \xi_k (-e^\sigma)^{k-1} A_0^{m-k}. \quad (3.21)$$

With all these data, we can finally compute the corrections to the stress tensor and charged currents and they take the following form,

$$\begin{aligned} \delta T_{00} &= 0, \quad \delta T^{ij} = 0, \quad \delta \tilde{J}_0 = 0 \\ \delta T_0^i &= -e^\sigma (\epsilon + p) \epsilon^{ijk\dots} \sum_{m=1}^n U_m(\sigma, A_0) (da)^{m-1} (dA)^{n-m} \\ \delta J_{Cov}^i &= \epsilon^{ijk\dots} \sum_{m=1}^n (J_m(\sigma, A_0) + q U_m(\sigma, A_0)) (da)^{m-1} (dA)^{n-m} \end{aligned} \quad (3.22)$$

Comparing the expressions for various components of stress tensor and covariant current of the fluid obtained from equilibrium partition function (3.15), (3.14) and fluid constitutive relations (3.22), we get,

$$\begin{aligned} U_m &= -\frac{e^{-2\sigma}}{\epsilon + p} [m\alpha_m - (n - m + 1)A_0\alpha_{m-1}] \\ &= -\frac{e^{-2\sigma}}{\epsilon + p} \left[ m\tilde{C}_m T_0^{m+1} - (n + 1 - m)\tilde{C}_{m-1} A_0 T_0^m \right. \\ &\quad \left. + \binom{n+1}{m+1} \mathcal{C}_{anom} A_0^{m+1} \right] \end{aligned} \quad (3.23)$$

Similarly, we can evaluate  $J_m(\sigma, A_0)$  as follows,

$$\begin{aligned}
J_m = e^{-\sigma} & \left[ -(m+1)\mathcal{C}_{anom}A_0^m \binom{n+1}{m+1} + (n-m+1)\tilde{C}_{m-1}T_0^m \right] \\
& + \frac{qe^{-2\sigma}}{\epsilon+p} \left[ m\tilde{C}_mT_0^{m+1} - (n+1-m)\tilde{C}_{m-1}A_0T_0^m \right. \\
& \quad \left. + \binom{n+1}{m+1}\mathcal{C}_{anom}A_0^{m+1} \right]
\end{aligned} \tag{3.24}$$

We want to now use this to obtain the transport coefficients  $\xi_m$  in the last relation of (3.17). For this we have to invert the relations (3.21) for  $\xi_m$ . We finally get

$$\begin{aligned}
\xi_m = & \left[ m\frac{q\mu}{\epsilon+p} - (m+1) \right] \mathcal{C}_{anom} \binom{n+1}{m+1} \mu^m \\
& + \sum_{k=0}^m \left[ m\frac{q\mu}{\epsilon+p} - (m-k) \right] (-1)^{k-1} \tilde{C}_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k-1}
\end{aligned} \tag{3.25}$$

This then is the prediction of this transport coefficient via partition function methods. This exactly matches with the expression from [14] in (2.18) provided we make the following identification among the constants  $\tilde{C}_m = (-1)^{m-1}C_m$ .

## 4. Comments on Most Generic Entropy Current

Another physical requirement which has long been used as a source of constraints on fluid dynamical transport coefficients is the local form of second law of thermodynamics. As we reviewed in the subsection§2.1 this principle had been used in [14] to obtain anomaly induced transports coefficients in arbitrary even dimensions.

In this section we will determine the entropy current in equilibrium by comparing the total entropy with that obtained from the equilibrium partition function. In the examples studied in [16, 17] it was seen that in general the comparison with equilibrium entropy ( obtained from partition function) did not fix all the non dissipative coefficients in fluid dynamical entropy current. However it did determine the anomalous contribution exactly. Here we will see that this holds true in general even dimensions.

Let us begin by computing the entropy from the equilibrium partition function. We begin with the anomalous part of the partition function

$$\begin{aligned}
W_{anom} = \frac{1}{T_0} \int d^{2n-1}x \sqrt{g_{2n-1}} & \left\{ \sum_{m=1}^n \alpha_{m-1} [\epsilon A(da)^{m-1}(dA)^{n-m}] \right. \\
& \left. + \alpha_n [\epsilon a(da)^{n-1}] \right\}
\end{aligned} \tag{4.1}$$

where the functions  $\alpha_m$  are given in (3.13).

The anomalous part of the total entropy is easily computed to be

$$\begin{aligned}
S_{anom} &= \frac{\partial}{\partial T_0} (T_0 W_{anom}) \\
&= \int d^{2n-1}x \sqrt{g_{2n-1}} \left\{ \sum_{m=1}^n m T_0^{m-1} \tilde{C}_{m-1} [\epsilon A (da)^{m-1} (dA)^{n-m}] \right. \\
&\quad \left. + (n+1) \tilde{C}_n T_0^n [\epsilon a (da)^{n-1}] \right\} \\
&= \int d^{2n-1}x \sqrt{g_{2n-1}} \left\{ \sum_{m=1}^n (m+1) T_0^m \tilde{C}_m [\epsilon a (da)^{m-1} (dA)^{n-m}] \right. \\
&\quad \left. + \tilde{C}_0 [\epsilon A (dA)^{n-1}] \right\}
\end{aligned} \tag{4.2}$$

Now we will determine the most general form of entropy current in equilibrium by comparison with (4.2). In [16] it was argued that the entropy current by itself is not a physical object, but entropy production and total entropy are. This gave a window for gauge non invariant contribution to entropy current but the contribution was removed by CPT invariance. Here also we will allow for such gauge non invariant terms in the entropy current. The most general form of entropy current, allowing for gauge non invariant pieces, is then

$$\begin{aligned}
J_S^\mu &= s u^\mu - \frac{\mu}{T} J_{odd}^\mu + \sum_{m=1}^n \chi_m \varepsilon^{\mu\nu\dots} u_\nu (\partial u)^{m-1} (\partial \hat{A})^{n-m} \\
&\quad + \zeta \varepsilon^{\mu\nu\dots} \hat{A}_\nu (\partial \hat{A})^{n-1}
\end{aligned} \tag{4.3}$$

where  $\chi_m$  is a function of  $T$  and  $\mu$  whereas  $\zeta$  is a constant. The correction to the local entropy density (i.e., the time component of the entropy current) can be written after an integration by parts as

$$\delta J_S^0 = \varepsilon^{0ij\dots} \left[ \zeta A (dA)^{n-1} + \sum_{k=1}^n \tilde{f}_k a (da)^{k-1} (dA)^{n-k} \right]_{ij\dots} + \text{total derivatives} \tag{4.4}$$

where

$$\tilde{f}_m \equiv -s U_m + \frac{\mu}{T} J_m + \zeta A_0^m \binom{n}{m} + \sum_{k=1}^m \binom{n-k}{m-k} \chi_k (-e^\sigma)^k A_0^{m-k} \tag{4.5}$$

The correction to the entropy is then,

$$\begin{aligned}
\delta S &= \int d^{2n-1}x \sqrt{g_{2n}} J_S^0 \\
&= \int d^{2n-1}x \sqrt{g_{2n-1}} \left[ \zeta [\epsilon A (dA)^{n-1}] + \sum_{m=1}^n \tilde{f}_m [\epsilon a (da)^{m-1} (dA)^{n-m}] \right]
\end{aligned} \tag{4.6}$$

Comparing the two expressions of total equilibrium entropy (4.2) and (4.6) we find the following expressions of the various coefficients in the entropy current (4.4),

$$\zeta = \tilde{C}_0 \quad \text{and} \quad \tilde{f}_k = (k+1) T_0^k \tilde{C}_k \quad \text{for } 0 \leq k \leq n \quad (4.7)$$

This in turn implies that

$$\begin{aligned} T_0 \sum_{k=1}^m \binom{n-k}{m-k} \chi_k (-e^\sigma)^k A_0^{m-k} \\ = \tilde{C}_m T_0^{m+1} + m \binom{n}{m} \mathcal{C}_{anom} A_0^{m+1} - \tilde{C}_0 T_0 A_0^m \binom{n}{m} \end{aligned} \quad (4.8)$$

which can be inverted to give

$$\begin{aligned} \chi_m = -\mathcal{C}_{anom} \binom{n+1}{m+1} T^{-1} \mu^{m+1} - \sum_{k=0}^m \tilde{C}_k (-1)^{k-1} \binom{n-k}{m-k} T^k \mu^{m-k} \\ \zeta = \tilde{C}_0 \end{aligned} \quad (4.9)$$

which matches with the prediction from [14] in equation (2.20) again with the identification  $C_m(-1)^{m-1} = \tilde{C}_m$ . We see that in the entropy current we have a total of  $n+1$  constants as in the equilibrium partition function.

This completes our partition function analysis and our re derivation of the results of [14] via partition function techniques. We see that the transport coefficients match exactly with the results obtained via entropy current (provided the analysis of [14] is extended by allowing gauge-non-invariant pieces in the entropy current). This detailed match of transport coefficients warrants the question whether the form of the equilibrium partition function itself can be directly derived from the expressions of [14] quoted in 2.1. We turn to this question in the next section.

## 5. Gibbs current and Partition function

We begin by repeating the expression for the Gibbs current in (2.10) which was central to the results of [14].

$$\begin{aligned} \bar{\mathcal{G}}_{anom}^{Cov} = C_0 T \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} + \sum_{m=1}^n \left[ \mathcal{C}_{anom} \binom{n+1}{m+1} \mu^{m+1} \right. \\ \left. + \sum_{k=0}^m C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k} \right] (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \end{aligned} \quad (5.1)$$

The subscript ‘anom’ denotes that we are considering only a part of the entropy current relevant to anomalies. The superscript ‘Cov’ refers to the fact that this is the Gibbs free energy computed by turning on a chemical potential for the **covariant** charge.

Let us ask how this expression would be modified if the Gibbs free energy was computed by turning on a chemical potential for the **consistent** charge instead. The change from covariant charge to consistent charge/current is simply given by a shift as given by the equation(2.9). This shift does not depend on the state of the theory but is purely a functional of the background gauge fields. Thinking of Gibbs free energy as minus temperature times the logarithm of the Euclidean path integral, a conversion from covariant charge to a consistent charge induces a shift

$$\bar{\mathcal{G}}_{anom}^{Cov} = \bar{\mathcal{G}}_{anom}^{Consistent} - \mu \, n \, \mathcal{C}_{anom} \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1}$$

which gives

$$\begin{aligned} & \bar{\mathcal{G}}_{anom}^{Consistent} \\ &= \sum_{m=1}^n \left[ \mathcal{C}_{anom} \binom{n+1}{m+1} \mu^{m+1} + \sum_{k=0}^m C_k \binom{n-k}{m-k} T^{k+1} \mu^{m-k} \right] (2\omega)^{m-1} \mathcal{B}^{n-m} \wedge u \quad (5.2) \\ & \quad + [C_0 T + n \mathcal{C}_{anom} \mu] \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} \end{aligned}$$

This now a Gibbs current whose  $\mu$  derivative gives the consistent current rather than a covariant current. It is easy to check that this solves an adiabaticity equation very similar to the one quoted in equation(2.4)

$$\begin{aligned} & d\bar{\mathcal{G}}_{anom}^{Consistent} + \mathfrak{a} \wedge \bar{\mathcal{G}}_{anom}^{Consistent} + n \mathcal{C}_{anom} (\hat{\mathcal{A}} + \mu u) \wedge \mathcal{E} \wedge \mathcal{B}^{n-1} \\ &= (dT + \mathfrak{a}T) \wedge \frac{\partial \bar{\mathcal{G}}_{anom}^{Consistent}}{\partial T} + (d\mu + \mathfrak{a}\mu - \mathcal{E}) \wedge \frac{\partial \bar{\mathcal{G}}_{anom}^{Consistent}}{\partial \mu} \quad (5.3) \end{aligned}$$

The question we wanted to address is how this Gibbs current is related to the partition function in equation (3.6).

The answer turns out to be quite intuitive - we would like to argue in this section that

$$W_{anom} = \ln Z_{Consistent}^{anom} = - \int_{space} \frac{1}{T} \bar{\mathcal{G}}_{anom}^{Consistent} \quad (5.4)$$

This equation instructs us to pull back the  $2n - 1$  form in equation (5.2) (divided by local temperature) and integrate it on an arbitrary spatial hyperslice to obtain the anomalous contribution to negative logarithm of the equilibrium path integral. Note that pulling back the Hodge dual of Gibbs current on a spatial hyperslice is essentially equivalent to integrating its zero component (i.e., the Gibbs density) on the slice. Seen this way the above relation is the familiar statement relating Gibbs free energy to the grand-canonical partition function.

## 5.1 Reproducing the Gauge variation

Before giving an explicit proof of the relation(5.4) we will check in this subsection that the relation(5.4) essentially gives the correct gauge variation to the path-integral at

equilibrium. This will provide us with a clearer insight on how the program of [16] to write a local expression in the partition function to reproduce the anomaly works.

The gauge variation of (5.4) under  $\delta\hat{\mathcal{A}} = d\delta\lambda$  is

$$\begin{aligned}
\delta W_{anom} &= \delta \ln Z_{Consistent}^{anom} = - \int_{space} \frac{1}{T} \delta \bar{\mathcal{G}}_{anom}^{Consistent} \\
&= - \int_{space} \left[ C_0 + n \mathcal{C}_{anom} \frac{\mu}{T} \right] \delta \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} \\
&= - \int_{space} \left[ C_0 + n \mathcal{C}_{anom} \frac{\mu}{T} \right] d\delta\lambda \wedge \mathcal{F}^{n-1} \\
&= - \int_{surface} \delta\lambda \left[ C_0 + n \mathcal{C}_{anom} \frac{\mu}{T} \right] \wedge \mathcal{F}^{n-1} + n \mathcal{C}_{anom} \int_{space} \delta\lambda d \left( \frac{\mu}{T} \right) \wedge \mathcal{F}^{n-1}
\end{aligned} \tag{5.5}$$

We will now ignore the surface contribution and use the fact that chemical equilibrium demands that

$$Td \left( \frac{\mu}{T} \right) = \mathcal{E}$$

where  $\mathcal{E} \equiv u^\nu \mathcal{F}_{\mu\nu} dx^\nu$  is the rest frame electric-field. This is essentially a statement (familiar from say semiconductor physics) that in equilibrium the diffusion current due to concentration gradients should cancel the drift ohmic current due to the electric field. Putting this in along with the electric-magnetic decomposition  $\mathcal{F} = \mathcal{B} + u \wedge \mathcal{E}$ , we get

$$\delta W_{anom} = \delta \ln Z_{Consistent}^{anom} = \mathcal{C}_{anom} \int_{space} \frac{\delta\lambda}{T} n \mathcal{E} \wedge \mathcal{B}^{n-1} \tag{5.6}$$

which is the correct anomalous variation required of the equilibrium path-integral ! In  $d = 2n = 4$  dimensions for example we get the correct  $E.B$  variation along with the  $1/T$  factor coming from the integration over euclidean time-circle. The factor of  $n$  comes from converting to electric and magnetic fields

$$\mathcal{F}^n = n u \wedge \mathcal{E} \wedge \mathcal{B}^{n-1}$$

Thus the shift piece along with the chemical equilibrium conspires to reproduce the correct gauge variation. The reader might wonder why this trick cannot be made to work by just keeping the shift term alone in the Gibbs current - the answer is of course that other terms are required if one insists on adiabaticity in the sense that we want to solve (5.3).

## 5.2 Integration by parts

In this subsection we will prove (5.4) explicitly. We will begin by evaluating the consistent Gibbs current in the equilibrium configuration. We will as before work in the ‘zero  $\mu_0$ ’ gauge.



Using the relations in the appendix B we get the consistent Gibbs current as

$$\begin{aligned}
& -\frac{1}{T} \bar{\mathcal{G}}_{anom}^{Consistent} \\
& = \frac{1}{T_0} \sum_{m=1}^n \left[ C_m (-1)^{m-1} T_0^{m+1} - C_0 (-1)^{0-1} \binom{n}{m} T_0 A_0^m \right. \\
& \quad \left. - \binom{n}{m+1} \mathcal{C}_{anom} A_0^{m+1} \right] (da)^{m-1} (dA)^{n-m} \wedge (dt + a) \\
& \quad - \frac{1}{T_0} [n \mathcal{C}_{anom} A_0 + C_0 T_0] A \wedge (dA + A_0 da)^{n-1} \\
& \quad - \frac{(n-1)}{T_0} [n \mathcal{C}_{anom} A_0 + C_0 T_0] A \wedge dA_0 \wedge (dt + a) \wedge (dA + A_0 da)^{n-2}
\end{aligned} \tag{5.7}$$

After somewhat long set of manipulations one arrives at the following form for the consistent Gibbs current

$$\begin{aligned}
& -\frac{1}{T} \bar{\mathcal{G}}_{anom}^{Consistent} \\
& = d \left\{ \frac{A}{T_0} \sum_{m=1}^{n-1} \left[ C_m (-1)^{m-1} T_0^{m+1} - C_0 (-1)^{0-1} \binom{n-1}{m} T_0 A_0^m \right. \right. \\
& \quad \left. \left. + m \binom{n}{m+1} \mathcal{C}_{anom} A_0^{m+1} \right] (da)^{m-1} (dA)^{n-1-m} \wedge (dt + a) \right\} \\
& \quad + \frac{A}{T_0} \sum_{m=1}^n \left[ C_{m-1} (-1)^{m-2} T_0^m - \binom{n}{m} \mathcal{C}_{anom} A_0^m \right] (da)^{m-1} (dA)^{n-m} \\
& \quad + C_n (-1)^{n-1} T_0^n (da)^{n-1} \wedge (dt + a)
\end{aligned} \tag{5.8}$$

Here we have taken out a surface contribution which we will suppress from now on since it does not contribute to the partition function. This final form is easily checked term by term and we will leave that as an exercise to the reader.

Suppressing the surface contribution we can write

$$\begin{aligned}
& -\frac{1}{T} \bar{\mathcal{G}}_{anom}^{Consistent} \\
& = d[\dots] + \frac{A}{T_0} \sum_{m=1}^n \left[ C_{m-1} (-1)^{m-2} T_0^m - \binom{n}{m} \mathcal{C}_{anom} A_0^m \right] (da)^{m-1} (dA)^{n-m} \\
& \quad + C_n (-1)^{n-1} T_0^n (da)^{n-1} \wedge (dt + a) \\
& = d[\dots] + \frac{A}{T_0} \wedge \sum_{m=1}^n \alpha_{m-1} (da)^{m-1} (dA)^{n-m} + \frac{dt + a}{T_0} \wedge \alpha_n (da)^{n-1}
\end{aligned} \tag{5.9}$$

where we have defined

$$\begin{aligned}
\alpha_m & = C_m (-1)^{m-1} T_0^{m+1} - \binom{n}{m+1} \mathcal{C}_{anom} A_0^{m+1} \quad \text{for } m < n \\
\alpha_n & = C_n (-1)^{n-1} T_0^{n+1}
\end{aligned} \tag{5.10}$$

To get the contribution to the equilibrium partition function, we integrate the above equation over the spatial slice (putting  $dt = 0$ ). We will neglect surface contributions to get

$$\begin{aligned}
& (\ln \mathcal{Z})_{anom}^{Consistent} \\
&= \int_{\text{space}} \frac{A}{T_0} \wedge \sum_{m=1}^n \left[ C_{m-1} (-1)^{m-2} T_0^m - \binom{n}{m} \mathcal{C}_{anom} A_0^m \right] (da)^{m-1} (dA)^{n-m} \\
&\quad + \int_{\text{space}} C_n (-1)^{n-1} T_0^n a \wedge (da)^{n-1} \\
&= \int_{\text{space}} \frac{A}{T_0} \wedge \sum_{m=1}^n \alpha_{m-1} (da)^{m-1} (dA)^{n-m} + \int_{\text{space}} \frac{a}{T_0} \wedge \alpha_n (da)^{n-1}
\end{aligned} \tag{5.11}$$

with  $\alpha_m$ s given by (5.10). We are essentially done - we have got the form in (3.6) and comparing the equations (5.10) and (3.13) we find a perfect agreement with the usual relation  $C_m (-1)^{m-1} = \tilde{C}_m$ . Now by varying this partition function we can obtain currents as before (the variation can be directly done in form language using the equations we provide in appendix C). With this we have completed a whole circle showing that the two formalisms for anomalous transport developed in [14] and [16] are completely equivalent.

Before we conclude, let us rewrite the partition function in terms of the polynomial  $\mathfrak{F}_{anom}^\omega[T, \mu]$  as

$$\begin{aligned}
& (\ln \mathcal{Z})_{anom}^{Consistent} \\
&= \int_{\text{space}} \frac{A}{T_0 da} \wedge \left[ \frac{\mathfrak{F}_{anom}^\omega[-T_0 da, dA] - \mathfrak{F}_{anom}^\omega[-T_0 da, 0]}{dA} - \frac{\mathfrak{F}_{anom}^\omega[0, dA + A_0 da]}{dA + A_0 da} \right] \\
&\quad + \int_{\text{space}} \frac{\mathfrak{F}_{anom}^\omega[-T_0 da, 0]}{(T_0 da)^2} \wedge T_0 a
\end{aligned} \tag{5.12}$$

We will consider an example. Using adiabaticity arguments, the authors of [8] derived the following expression for a theory of free Weyl fermions in  $d = 2n$  spacetime dimensions

$$(\mathfrak{F}_{anom}^\omega)_{d=2n}^{free \text{ Weyl}} = -2\pi \sum_{\text{species}} \chi_{d=2n} \left[ \frac{\frac{\tau}{2} T}{\sin \frac{\tau}{2} T} e^{\frac{\tau}{2\pi} q\mu} \right]_{\tau^{n+1}} \tag{5.13}$$

where  $\chi_{d=2n}$  is the chirality and the subscript  $\tau^{n+1}$  denotes that one needs to Taylor-expand in  $\tau$  and retain the coefficient of  $\tau^{n+1}$ . Substituting this into the above expression gives the anomalous part of the partition function of free Weyl fermions.

## 6. Fluids charged under multiple $U(1)$ fields

In this section, we will generalize our results to cases where we have multiple abelian  $U(1)$  gauge fields in arbitrary  $2n$ -dimensions.

We can take

$$\mathfrak{F}_{anom}^\omega[T, \mu] = \mathcal{C}_{anom}^{A_1 \dots A_{n+1}} \mu_{A_1} \dots \mu_{A_{n+1}} + \sum_{m=0}^n C_m^{A_1 \dots A_{n-m}} T^{m+1} \mu_{A_1 \dots A_{n-m}}. \quad (6.1)$$

In this case, the anomaly equation takes the following form,

$$\nabla_\mu J_{Cov}^{\mu, A_{n+1}} = \frac{n+1}{2^n} \mathcal{C}_{anom}^{A_1 A_2 \dots A_{n+1}} \varepsilon^{\mu_1 \nu_1 \mu_2 \nu_2 \dots \mu_n \nu_n} (\mathcal{F}_{\mu_1 \nu_1})_{A_1} \dots (\mathcal{F}_{\mu_n \nu_n})_{A_n}. \quad (6.2)$$

Where, in  $2n$  dimensions  $\mathcal{C}_{anom}$  has  $n+1$  indices denoted by  $(A_1, A_2 \dots A_{n+1})$  and it is symmetric in all its indices. It is straightforward to carry on the above computation for the case of multiple  $U(1)$  charges and most of the computations remains the same. Now, for the multiple  $U(1)$  case, in partition function 3.6 the functions  $\alpha_m$  and the constants  $\tilde{C}_m$  (and the constants  $C_m$  appearing in  $\mathfrak{F}_{anom}^\omega$ ) have  $n-m$  number of indices which are contracted with  $n-1-m$  number of  $dA$  and one  $A$ . The constant  $\zeta$  appearing in the entropy current has  $n$  indices.

The constant  $\tilde{C}_n$  (and  $\alpha_n$ ) has no index. All these constants are symmetric in their indices. Considering the above index structure into account, we can understand that the functions  $U_m$  appearing in velocity correction and  $\chi_m$  appearing in entropy corrections has  $n-m$  indices and the function  $J_m$  appearing in the charge current has  $n-m+1$  indices. Now, we can write the generic form of these functions as follows:

$$\begin{aligned} U_m^{A_1 A_2 \dots A_{n-m}} = & -\frac{e^{-2\sigma}}{\epsilon + p} \left[ m \tilde{C}_m^{A_1 A_2 \dots A_{n-m}} T_0^{m+1} \right. \\ & - (n+1-m) \tilde{C}_{m-1}^{A_1 A_2 \dots A_{n-m} B_1} (A_0)_{B_1} T_0^m \\ & \left. + \binom{n+1}{m+1} \mathcal{C}_{anom}^{A_1 \dots A_{n-m} B_1 \dots B_{m+1}} (A_0)_{B_1} \dots (A_0)_{B_{m+1}} \right] \end{aligned} \quad (6.3)$$

where  $(A_0)_{B_1}$  comes from the  $B_1$ th gauge field.

Similarly, we can write the coefficients appearing in  $A$ 'th charge current  $(J^A)$  as,

$$\begin{aligned} (J^A)_m^{A_1 A_2 \dots A_{n-m}} = & e^{-\sigma} \left[ -(m+1) \mathcal{C}_{anom}^{A A_1 \dots A_{n-m} B_1 \dots B_m} (A_0)_{B_1} \dots (A_0)_{B_m} \binom{n+1}{m+1} \right. \\ & \left. + (n-m+1) \tilde{C}_{m-1}^{A A_1 \dots A_{n-m}} T_0^m \right] \\ & + \frac{q^A e^{-2\sigma}}{\epsilon + p} \left[ m \tilde{C}_m^{A_1 A_2 \dots A_{n-m}} T_0^{m+1} \right. \\ & - (n+1-m) \tilde{C}_{m-1}^{A_1 A_2 \dots A_{n-m} B_1} (A_0)_{B_1} T_0^m \\ & \left. + \binom{n+1}{m+1} \mathcal{C}_{anom}^{A_1 \dots A_{n-m} B_1 \dots B_{m+1}} (A_0)_{B_1} \dots (A_0)_{B_{m+1}} \right] \end{aligned} \quad (6.4)$$

Name	Symbol	CPT
Temperature	$T$	+
Chemical Potential	$\mu$	-
Velocity 1-form	$u$	+
Gauge field 1-form	$\hat{\mathcal{A}}$	-
Exterior derivative	$d$	-
Field strength 2-form	$\mathcal{F} = d\hat{\mathcal{A}}$	+
Magnetic field 2-form	$\mathcal{B}$	+
Vorticity 2-form	$\omega$	-

**Table 1:** Action of CPT on various forms

We can also express the transport coefficients for fluids charged under multiple  $U(1)$  charges, generalising equation (3.25) as,

$$\begin{aligned}
& (\xi^A)_{\mathbf{m}}^{A_1 A_2 \dots A_{n-m}} \\
&= \left[ m \frac{q^A \mu_B}{\epsilon + p} - (m+1) \delta_B^A \right] \mathcal{C}_{anom}^{BA_1 \dots A_{n-m} B_1 \dots B_m} \binom{n+1}{m+1} \mu_{B_1} \dots \mu_{B_m} \\
&+ \sum_{k=0}^{m-1} \left[ m \frac{q^A \mu_B}{\epsilon + p} - (m-k) \delta_B^A \right] \\
&\quad \times (-1)^{k-1} \tilde{C}_k^{BA_1 \dots A_{n-m} B_1 \dots B_{m-k-1}} \binom{n-k}{m-k} T^{k+1} \mu_{B_1} \dots \mu_{B_{m-k-1}} \\
&+ \left[ m \frac{q^A}{\epsilon + p} \right] (-1)^{m-1} \tilde{C}_m^{A_1 \dots A_{n-m}} T^{m+1}
\end{aligned} \tag{6.5}$$

Similarly the coefficients  $\chi_m$  appearing entropy current become

$$\begin{aligned}
\chi_m^{A_1 \dots A_{n-m}} &= -\mathcal{C}_{anom}^{A_1 \dots A_{n-m} B_1 \dots B_{m+1}} \binom{n+1}{m+1} T^{-1} \mu_{B_1} \dots \mu_{B_{m+1}} \\
&- \sum_{k=0}^m (-1)^{k-1} \binom{n-k}{m-k} T^k \tilde{C}_k^{A_1 \dots A_{n-m} B_1 \dots B_{m-k}} \mu_{B_1} \dots \mu_{B_{m-k}}
\end{aligned} \tag{6.6}$$

This finishes the analysis of anomalous fluid charged under multiple abelian  $U(1)$  gauge fields.

## 7. CPT Analysis

In this section we analyze the constraints of 2n dimensional CPT invariance on the analysis of our previous sections.

Let us first examine the CPT transformation of the Gibbs current proposed in [14]. Using the Table 1 we see that the Gibbs current in Eqn.(2.10) is CPT-even provided

the coefficients  $\{\mathcal{C}_{anom}, C_{2k+1}\}$  are CPT-even and the coefficients  $C_{2k}$  are CPT-odd. Since in a CPT-invariant theory all CPT-odd coefficients should vanish, we conclude that  $C_m = 0$  for even  $m$ . This conclusion can be phrased as

$$CPT \quad : \quad C_m(-1)^{m-1} = C_m \quad (7.1)$$

Note that this is the same conclusion as reached by assuming the relation to the anomaly polynomial.

Next we analyze the constraints of  $2n$  dimensional CPT invariance on the partition function (3.6). Our starting point is a partition function of the fluid and we expect it to be invariant under  $2n$  dimensional CPT transformation of the fields. Table 2 lists the effect of  $2n$  dimensional C, P and T transformation on various field appearing in the partition function (3.6). Since  $a_i$  is even while  $A_i$  and  $\partial_j$  are odd under CPT, the term with coefficient  $C_m$  picks up a factor of  $(-1)^{(m+1)}$ . Thus CPT invariance tells us that  $C_m$  must be

- even function of  $A_0$  for odd  $m$ .
- odd function of  $A_0$  for even  $m$ .

Now the coefficients  $C_m$  are fixed upto constants  $\tilde{C}_m$  by the requirement that the partition function reproduces the correct anomaly. Note that the  $A_0$  (odd under CPT) dependence of the coefficients  $C_m$  thus determined are consistent with the requirement CPT invariance. Further, CPT invariance forces  $\tilde{C}_m = 0$  for even  $m$ . The last term in the partition function (3.6) is odd under parity and thus its coefficient is set to zero by CPT for even  $n$  whereas for odd  $n$  it is left unconstrained.

Thus finally we see that CPT invariance allows for a total of

- $\frac{n}{2}$  constant ( $\tilde{C}_m$  with  $m$  odd) for even  $n$ .
- $\frac{n+1}{2}$  constants ( $\tilde{C}_m$  with  $m$  even and  $\tilde{C}_n$ ) for odd  $n$ .

In particular the coefficient  $\tilde{C}_0$  always vanishes and thus, for a CPT invariant theory, we never get the gauge-non invariant contribution to the local entropy current.

## 8. Conclusion

In this paper we have shown that the results of [15, 14] can be based on entropy arguments can be re derived within a more field-theory friendly partition function technique [16, 17, 19, 20]. This has led us to a deeper understanding linking the local description of anomalous transport in terms of a Gibbs current [14, 8] to the global description in terms of partition functions.

fields	C	P	T	CPT
$\sigma$	+	+	+	+
$a_i$	+	-	-	+
$g_{ij}$	+	+	+	+
$A_0$	-	+	+	-
$A_i$	-	-	-	-

**Table 2:** Action of CPT on various field

An especially satisfying result is that the polynomial structure of anomalous transport coefficients discovered in [14] is reproduced at the level of partition functions. There it was shown that the whole set of anomalous transport coefficients are essentially governed by a single homogeneous polynomial  $\mathfrak{F}_{anom}^\omega[T, \mu]$  of temperature and chemical potentials. The authors of [8] noticed that in a free theory of chiral fermions this polynomial structure is directly linked to the corresponding anomaly polynomial of chiral fermions via a replacement rule

$$\mathfrak{F}_{anom}^\omega[T, \mu] = \mathcal{P}_{anom} [\mathcal{F} \mapsto \mu, p_1(\mathfrak{R}) \mapsto -T^2, p_{k>1}(\mathfrak{R}) \mapsto 0] \quad (8.1)$$

This result could be generalised for an arbitrary free theory with chiral fermions and chiral p-form fields using sphere partition function techniques which link this polynomial to a specific thermal observable[22].

Various other known results (for example in AdS/CFT) support the conjecture that this rule is probably true in all theories with some mild assumptions. While we have succeeded in reproducing the polynomial structure we have not tried in this paper to check the above conjecture - this necessarily involves a similar analysis keeping track of the effect of gravitational anomalies which we have ignored in our work. It would be interesting to extend our analysis to theories with gravitational anomalies<sup>10</sup>.

We have derived in this paper a particular contribution to the equilibrium partition function that is linked to the underlying anomalies of the theory. A direct test of this result would be to do a direct holographic computation of the same quantity in AdS/CFT to obtain these contributions. Since the CFT anomalies are linked to the Chern-Simons terms in the bulk the holographic test would be a computation of a generalised Wald entropy for a black hole solution of a gravity theory with Chern-Simons terms. The usual Wald entropy gets modified in the presence of such Chern-Simons terms[24, 25] which are usually a part of higher derivative corrections to gravity. We hope that reproducing the results of this paper would give us a test of generalised Wald formalism for such higher derivative corrections.

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<sup>10</sup>As we were finalising this manuscript, a paper[23] dealing with 1 + 1d gravitational anomalies appeared in arXiv. We thank Amos Yarom for various discussions regarding this topic.

We have directly linked the description in terms of a Gibbs current[14, 8] satisfying a kind of adiabaticity equation to the global description in terms of partition functions. Further we have noticed in (2.21) that at least in the case of anomalous transport this Gibbs current is closely linked to what has been called ‘the non-canonical part of the entropy current ’ in various entropy arguments[26]. It would be interesting to see whether this construction can be generalised beyond the anomalous transport coefficients to other partition function computations which appear in [16, 19]. This would give us a more local interpretation of the various terms appearing in the partition function linking them to a specific Gibbs free energy transport process. Hence with such a result one could directly identify the coefficients appearing in the partition function as the transport coefficients of the Gibbs current.

Another interesting observation of [14] apart from the polynomial structure is that the anomalous transport satisfies an interesting reciprocity type relation (2.15) - the susceptibility describing the change in the anomalous charge current with a small change in vorticity is equal to the susceptibility describing the change in the anomalous energy current with a small change in magnetic field. While we see that the results of our paper are consistent with this observation made in [14], we have not succeeded in deriving this relation directly from the partition function. It would be interesting to derive such a relation from the partition function hence clarifying how such a relation arises in a microscopic description .

Finally as we have emphasised in the introductions one would hope that the results of our paper serve as a starting point for generalising the analysis of anomalies to non-equilibrium phenomena. Can one write down a Schwinger-Keldysh functional which transforms appropriately - does this provide new constraints on the dissipative transport coefficients ? We leave such questions to future work.

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## APPENDICES

### A. Results of $(3+1)$ – dimensional and $(1+1)$ – dimensional fluid

In this appendix we want to specialise our results to  $1+1$  and  $3+1$  dimensional anomalous fluids. By considering local entropy production of the system, the results for  $(3+1)$ – dimensional anomalous fluid were obtained in [12], [13, 27] and for  $(1+1)$ –dimensional fluid were obtained in [28]. The same results have also been obtained in [16] and [17] for  $(3+1)$ – dimensional and  $(1+1)$ –dimensional anomalous fluid respectively, by writing the equilibrium partition function, the technique that we have followed in this paper. Our goal in this section is to check that the arbitrary dimension results reduce correctly to these special cases.

#### A.1 $(3+1)$ – dimensional anomalous fluids

Let us consider fluid living in  $(3+1)$ –dimension and is charged under a  $U(1)$  current. Take

$$\mathfrak{F}_{anom}^\omega[T, \mu] = \mathcal{C}_{anom}^{d=4} \mu^3 + C_0^{d=4} T \mu^2 + C_1^{d=4} T^2 \mu + C_2^{d=4} T^2 \mu \quad (\text{A.1})$$

the constants  $\{C_0^{d=4}, C_2^{d=4}\}$  if non-zero violate CPT since their subscript indices are even.

By the replacement rule of [8] this corresponds to a theory with the anomaly polynomial

$$\mathcal{P}_{anom} = \mathcal{C}_{anom}^{d=4} \mathcal{F}^3 - C_1^{d=4} p_1(\mathfrak{R}) \wedge \mathcal{F} \quad (\text{A.2})$$

where  $p_1(\mathfrak{R})$  is the first-pontryagin 4-form of curvature.

We have

$$\begin{aligned} d\bar{J}_{Consistent} &= \mathcal{C}_{anom}^{d=4} \mathcal{F}^2 \\ d\bar{J}_{Cov} &= 3\mathcal{C}_{anom}^{d=4} \mathcal{F}^2 \end{aligned}$$

and their difference is given by

$$\bar{J}_{Cov} = \bar{J}_{Consistent} + 2\mathcal{C}_{anom}^{d=4} \hat{\mathcal{A}} \wedge \mathcal{F}$$

In components we have

$$\begin{aligned} \nabla_\mu J_{Consistent}^\mu &= \mathcal{C}_{anom}^{d=4} \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu} \mathcal{F}_{\rho\sigma}, \\ \nabla_\mu J_{Cov}^\mu &= 3\mathcal{C}_{anom}^{d=4} \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu} \mathcal{F}_{\rho\sigma}, \\ J_{Cov}^\mu &= J_{Consistent}^\mu + 2\mathcal{C}_{anom}^{d=4} \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \hat{\mathcal{A}}_\nu \mathcal{F}_{\rho\sigma} \end{aligned} \quad (\text{A.3})$$



The anomaly-induced transport coefficients (in Landau frame) in this case are given by

$$\begin{aligned}
J_{Cov}^{\mu,anom} &= \xi_1^{d=4} \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho \hat{\mathcal{A}}_\sigma + \xi_2^{d=4} \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma \\
\xi_1^{d=4} &= 3\mathcal{C}_{anom}^{d=4} \mu \left[ \frac{q\mu}{\epsilon + p} - 2 \right] + 2C_0^{d=4} T \left[ \frac{q\mu}{\epsilon + p} - 1 \right] + C_1^{d=4} T^2 \mu^{-1} \left[ \frac{q\mu}{\epsilon + p} \right] \\
\xi_2^{d=4} &= \mathcal{C}_{anom}^{d=4} \mu^2 \left[ 2 \frac{q\mu}{\epsilon + p} - 3 \right] + C_0^{d=4} T \mu \left[ 2 \frac{q\mu}{\epsilon + p} - 2 \right] \\
&\quad + C_1^{d=4} T^2 \mu \left[ 2 \frac{q\mu}{\epsilon + p} - 1 \right] + C_2^{d=4} T^3 \mu^{-1} \left[ 2 \frac{q\mu}{\epsilon + p} \right]
\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}
J_S^{\mu,anom} &= -\frac{\mu}{T} J_{Cov}^{\mu,anom} + \chi_1^{d=4} \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho \hat{\mathcal{A}}_\sigma + \chi_2^{d=4} \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma + \zeta^{d=4} \varepsilon^{\mu\nu\rho\sigma} \hat{\mathcal{A}}_\nu \partial_\rho \hat{\mathcal{A}}_\sigma \\
\mathcal{G}_{Cov}^{\mu,anom} &= -T \chi_1^{d=4} \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho \hat{\mathcal{A}}_\sigma - T \chi_2^{d=4} \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma - T \zeta^{d=4} \varepsilon^{\mu\nu\rho\sigma} \hat{\mathcal{A}}_\nu \partial_\rho \hat{\mathcal{A}}_\sigma \\
-\zeta^{d=4} &= C_0^{d=4} \\
-\chi_1^{d=4} &= 3\mathcal{C}_{anom}^{d=4} T^{-1} \mu^2 + 2C_0^{d=4} \mu + C_1^{d=4} T \\
-\chi_2^{d=4} &= \mathcal{C}_{anom}^{d=4} T^{-1} \mu^3 + C_0^{d=4} \mu^2 + C_1^{d=4} T \mu + C_2^{d=4} T^2
\end{aligned} \tag{A.5}$$

The anomalous part of the consistent partition function is given by

$$\begin{aligned}
(\ln \mathcal{Z})_{anom}^{Consistent} &= \int_{\text{space}} \frac{A}{T_0} \wedge \{ [C_0^{d=4}(-1)T_0 - 2\mathcal{C}_{anom}^{d=4}A_0] (dA) + [C_1^{d=4}T_0^2 - \mathcal{C}_{anom}^{d=4}A_0^2] (da) \} \\
&\quad + \int_{\text{space}} C_2^{d=4}(-1)T_0^2 a \wedge (da) \\
&= -\frac{\mathcal{C}_{anom}^{d=4}}{T_0} \int d^3x \sqrt{g_3} \epsilon^{ijk} [2A_0 A_i \partial_j A_k + A_0^2 A_i \partial_j a_k] \\
&\quad - C_0^{d=4} \int d^3x \sqrt{g_3} \epsilon^{ijk} A_i \partial_j A_k + C_1^{d=4} T_0 \int d^3x \sqrt{g_3} \epsilon^{ijk} A_i \partial_j a_k \\
&\quad - C_2^{d=4} T_0^2 \int d^3x \sqrt{g_3} \epsilon^{ijk} a_i \partial_j a_k
\end{aligned} \tag{A.6}$$

The results for the equilibrium partition function and the transport coefficients of the fluid have been obtained in [16] in great detail. We will now compare the results above against the results there. We begin by first fixing the relation between the notation here and the notation employed in [16]. Comparing our partition function in (A.6) against Eqn(1.11) of [16] we get a perfect match with the following relabeling of constants<sup>11</sup>

$$\mathcal{C}_{anom}^{d=4} = \frac{C}{6}, \quad C_0^{d=4} = -C_0, \quad C_1^{d=4} = C_2, \quad C_2^{d=4} = -C_1 \tag{A.7}$$

---

<sup>11</sup>We warn the reader that the wedge notation in [16] differs from the one we use by numerical factors. So the comparisons are to be made *after* converting to explicit components to avoid confusion.

The first of these relations also follows independently from comparing our eqn(A.3) against the corresponding equations in [16] for covariant/consistent anomaly and the Bardeen current. We then proceed to compare the transport coefficients in Eqn(3.12) and Eqn.(3.21) of [16] against our results in (A.4) and (A.5).

We get a match provided one uses (in addition to (A.7) ) the following relations arising from comparing definitions here against [16]

$$\xi_B = \xi_1^{d=4}, \quad \xi_\omega = 2\xi_2^{d=4}, \quad D_B = \chi_1^{d=4}, \quad D_\omega = 2\chi_2^{d=4}, \quad h = \zeta^{d=4} \quad (\text{A.8})$$

## A.2 (1 + 1)– dimensional anomalous fluids

Let us consider fluid living in (1 + 1)–dimension and is charged under a  $U(1)$  current. Take

$$\mathfrak{F}_{anom}^\omega[T, \mu] = \mathcal{C}_{anom}^{d=2} \mu^2 + C_0^{d=2} T \mu + C_1^{d=2} T^2 \quad (\text{A.9})$$

the constant  $C_0^{d=2}$  if non-zero violates CPT since its subscript index is even.

By the replacement rule of [8] this corresponds to a theory with the anomaly polynomial

$$\mathcal{P}_{anom} = \mathcal{C}_{anom}^{d=2} \mathcal{F}^2 - C_1^{d=2} p_1(\mathfrak{R}) \quad (\text{A.10})$$

where  $p_1(\mathfrak{R})$  is the first-pontryagin 4-form of curvature.

We have

$$\begin{aligned} d\bar{J}_{Consistent} &= \mathcal{C}_{anom}^{d=2} \mathcal{F} \\ d\bar{J}_{Cov} &= 2\mathcal{C}_{anom}^{d=2} \mathcal{F} \end{aligned}$$

and their difference is given by

$$\bar{J}_{Cov} = \bar{J}_{Consistent} + \mathcal{C}_{anom}^{d=2} \hat{\mathcal{A}}$$

In components we have

$$\begin{aligned} \nabla_\mu J_{Consistent}^\mu &= \mathcal{C}_{anom}^{d=2} \frac{1}{2} \varepsilon^{\mu\nu} \mathcal{F}_{\mu\nu}, \\ \nabla_\mu J_{Cov}^\mu &= 2\mathcal{C}_{anom}^{d=2} \frac{1}{2} \varepsilon^{\mu\nu} \mathcal{F}_{\mu\nu}, \\ J_{Cov}^\mu &= J_{Consistent}^\mu + \mathcal{C}_{anom}^{d=2} \varepsilon^{\mu\nu} \hat{\mathcal{A}}_\nu \end{aligned} \quad (\text{A.11})$$

The anomaly-induced transport coefficients (in Landau frame) in this case are given by

$$\begin{aligned} J_{Cov}^{\mu, anom} &= \xi_1^{d=2} \varepsilon^{\mu\nu} u_\nu \\ \xi_1^{d=2} &= \mathcal{C}_{anom}^{d=2} \mu \left[ \frac{q\mu}{\epsilon + p} - 2 \right] + C_0^{d=2} T \left[ \frac{q\mu}{\epsilon + p} - 1 \right] + C_1^{d=2} T^2 \mu^{-1} \left[ \frac{q\mu}{\epsilon + p} \right] \end{aligned} \quad (\text{A.12})$$

and

$$\begin{aligned}
J_S^{\mu,anom} &= -\frac{\mu}{T} J_{Cov}^{\mu,anom} + \chi_1^{d=2} \varepsilon^{\mu\nu} u_\nu + \zeta^{d=2} \varepsilon^{\mu\nu} \hat{A}_\nu \\
\mathcal{G}_{Cov}^{\mu,anom} &= -T \chi_1^{d=2} \varepsilon^{\mu\nu} u_\nu - T \zeta^{d=2} \varepsilon^{\mu\nu} \hat{A}_\nu \\
-\zeta^{d=2} &= C_0^{d=2} \\
-\chi_1^{d=2} &= \mathcal{C}_{anom}^{d=2} T^{-1} \mu^2 + C_0^{d=2} \mu + C_1^{d=2} T
\end{aligned} \tag{A.13}$$

The anomalous part of the consistent partition function is given by

$$\begin{aligned}
&(\ln \mathcal{Z})_{anom}^{Consistent} \\
&= \int_{\text{space}} \frac{A}{T_0} \wedge [C_0^{d=2} (-1) T_0 - \mathcal{C}_{anom}^{d=2} A_0] + \int_{\text{space}} C_1^{d=2} T_0 a \\
&= -\frac{\mathcal{C}_{anom}^{d=2}}{T_0} \int dx \sqrt{g_1} \epsilon^i A_0 A_i - C_0^{d=2} \int dx \sqrt{g_1} \epsilon^i A_i + C_1^{d=2} T_0 \int dx \sqrt{g_1} \epsilon^i a_i
\end{aligned} \tag{A.14}$$

Now we are all set to compare our results with the results of [17]. The comparison proceeds here the same way as the comparison in 3+1d before. By comparing Eqn(2.4) of [17] against our (A.14) we get<sup>12</sup>

$$C_{anom}^{d=2} = C, \quad C_0^{d=2} = -C_1, \quad C_1^{d=2} = -C_2, \tag{A.15}$$

and we get a match of transport coefficients using the definitions

$$\xi_j = \xi_1^{d=2}, \quad \xi_s + \frac{\mu}{T} \xi_j = \chi_1^{d=2}, \quad D_\omega = 2\chi_2^{d=4}, \quad h = \zeta^{d=2} \tag{A.16}$$

## B. Hydrostatics and Anomalous transport

In this section we will follow [16, 17] in describing a hydrostatic configuration, i.e., a time-independent hydrodynamic configuration in a gauge/gravitational background. We will then proceed to evaluate the anomalous currents derived in previous section in this background. This is followed by a computation of consistent partition function by integrating the consistent Gibbs current over a spatial slice. For convenience we will phrase our entire discussion in the language of forms (as in the previous section) and refer the reader to the appendixD for our form conventions.

Let us consider the special case where we consider a stationary (time-independent) spacetime with a metric given by

$$g_{spacetime} = -\gamma^{-2} (dt + a)^2 + g_{space}$$

---

<sup>12</sup>Note that authors of [17] set the CPT-violating coefficient  $C_0^{d=2} = -C_1 = 0$  in most of their analysis. This fact has to be accounted for during the comparison.

where in the notation of [16] we can write  $\gamma \equiv e^{-\sigma}$ . Following the discussion there, consider a time-independent fluid configuration with local temperature and chemical potential  $T, \mu$  and

placed in a time-independent gauge-field background

$$\hat{\mathcal{A}} = \mathcal{A}_0 dt + \mathcal{A}$$

We first compute

$$\begin{aligned}\mathcal{E} &\equiv \mathcal{F}_{\mu\nu} dx^\mu u^\nu = \gamma \mathcal{F}_{i0} dx^i = \gamma d\mathcal{A}_0 \\ \mathfrak{a} &\equiv u^\mu \nabla_\mu u_\nu dx^\nu = -\gamma^{-1} d\gamma = \gamma d\gamma^{-1} \\ dT + \mathfrak{a}T &= \gamma d(\gamma^{-1}T) \\ d\mu + \mathfrak{a}\mu - \mathcal{E} &= \gamma d(\gamma^{-1}\mu - \mathcal{A}_0)\end{aligned}\tag{B.1}$$

If we insist that

$$\begin{aligned}dT + \mathfrak{a}T &= 0 \\ d\mu + \mathfrak{a}\mu - \mathcal{E} &= 0\end{aligned}\tag{B.2}$$

then it follows that the quantities

$$T_0 \equiv \gamma^{-1}T \quad \text{and} \quad \mu_0 \equiv \gamma^{-1}\mu - \mathcal{A}_0$$

are constant across space. We can invert this to write

$$T = \gamma T_0 \quad \text{and} \quad \mu = \gamma(\mathcal{A}_0 + \mu_0) \equiv \gamma A_0$$

where we have defined  $A_0 \equiv \mathcal{A}_0 + \mu_0$ . Following [16] we will split the gauge field as

$$\hat{\mathcal{A}} = \mathcal{A}_0 dt + \mathcal{A} = A_0(dt + a) + A - \mu_0 dt$$

where  $A \equiv \mathcal{A} - A_0 a$ . We are now working in a general gauge - often it is useful to work in a specific gauge : one gauge we will work on is obtained from this generic gauge by performing a gauge transformation to remove the  $\mu_0 dt$  piece. We will call this gauge as the ‘zero  $\mu_0$ ’ gauge. In this gauge the new gauge field is given in terms of the old gauge field via

$$\hat{\mathcal{A}}_{\mu_0=0} \equiv \hat{\mathcal{A}} + \mu_0 dt$$

We will quote all our consistent currents in this gauge.

We are now ready to calculate various hydrostatic quantities

$$\begin{aligned}
\mathcal{E} &= \gamma d\mathcal{A}_0 = \gamma dA_0 \\
\mathfrak{a} &= -\gamma^{-1} d\gamma = \gamma d\gamma^{-1} \\
\mathcal{B} &\equiv \mathcal{F} - u \wedge \mathcal{E} = d[A_0(dt + a) + A - \mu_0 dt] + (dt + a) \wedge dA_0 \\
&= dA + A_0 da \\
2\omega &= du + u \wedge \mathfrak{a} = -\gamma^{-1} da \\
2\omega T &= -T_0 da \\
2\omega \mu &= -A_0 da \\
\hat{A} + \mu u &= A - \mu_0 dt \\
\mathcal{B} + 2\omega \mu &= dA
\end{aligned} \tag{B.3}$$

Now let us compute the various anomalous currents in terms of the hydrostatic fields. Using (B.3) we get the Gibbs current as

$$\begin{aligned}
& -\bar{\mathcal{G}}_{anom}^{Cov} \\
&= \gamma \sum_{m=1}^n \left[ C_m (-1)^{m-1} T_0^{m+1} - C_0 (-1)^{0-1} \binom{n}{m} T_0 A_0^m \right. \\
&\quad \left. + m \binom{n+1}{m+1} \mathcal{C}_{anom} A_0^{m+1} \right] (da)^{m-1} (dA)^{n-m} \wedge (dt + a) \\
&\quad - \gamma C_0 T_0 \hat{\mathcal{A}}_{\mu_0=0} \wedge \mathcal{F}^{n-1}
\end{aligned} \tag{B.4}$$

In the following we will always write the minus signs in the form  $C_m(-1)^{m-1}$  so that once we impose CPT all the minus signs could be dropped.

We can now calculate the charge/entropy/energy currents

$$\begin{aligned}
\bar{J}_{anom}^{Cov} &= \sum_{m=1}^n \left[ -(n+1-m) C_{m-1} (-1)^{m-2} T_0^m \right. \\
&\quad \left. + (n+1) \binom{n}{m} \mathcal{C}_{anom} A_0^m \right] (da)^{m-1} \wedge (dA)^{n-m} \wedge (dt + a)
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
\bar{J}_{S,anom}^{Cov} &= \sum_{m=1}^n \left[ (m+1) C_m (-1)^{m-1} T_0^m \right. \\
&\quad \left. - C_0 (-1)^{0-1} \binom{n}{m} A_0^m \right] (da)^{m-1} (dA)^{n-m} \wedge (dt + a) \\
&\quad - C_0 \hat{\mathcal{A}}_{\mu_0=0} \wedge \mathcal{F}^{n-1}
\end{aligned} \tag{B.6}$$

and

$$\begin{aligned}
\bar{q}_{anom}^{Cov} &= \gamma \sum_{m=1}^n \left[ m C_m (-1)^{m-1} T_0^{m+1} - (n+1-m) C_{m-1} (-1)^{m-2} T_0^m A_0 \right. \\
&\quad \left. + \binom{n+1}{m+1} \mathcal{C}_{anom} A_0^{m+1} \right] (da)^{m-1} (dA)^{n-m} \wedge (dt+a)
\end{aligned} \tag{B.7}$$

We can go to the Landau frame as before

$$\begin{aligned}
u^\mu &\mapsto u^\mu - \frac{q_{anom}^\mu}{\epsilon + p} \\
J_{anom}^\mu &\mapsto J_{anom}^\mu - q \frac{q_{anom}^\mu}{\epsilon + p} \\
J_{S,anom}^\mu &\mapsto J_{S,anom}^\mu - s \frac{q_{anom}^\mu}{\epsilon + p} \\
q_{anom}^\mu &\mapsto 0
\end{aligned} \tag{B.8}$$

In the Landau frame we can write the corrections to various quantities as

$$\begin{aligned}
\delta \bar{u} &\equiv -\gamma^{-1} \sum_{m=1}^n U_m (da)^{m-1} \wedge (dA)^{n-m} \wedge (dt+a) \\
\delta \bar{J}_{anom}^{Cov} &\equiv -\gamma^{-1} \sum_{m=1}^n (J_m + q U_m) (da)^{m-1} \wedge (dA)^{n-m} \wedge (dt+a) \\
\delta \bar{J}_{S,anom}^{Cov} &\equiv -\gamma^{-1} \sum_{m=1}^n (S_m + s U_m) (da)^{m-1} \wedge (dA)^{n-m} \wedge (dt+a)
\end{aligned} \tag{B.9}$$

where

$$\begin{aligned}
U_m &= -\frac{\gamma^2}{\epsilon + p} \left[ m C_m (-1)^{m-1} T_0^{m+1} - (n+1-m) C_{m-1} (-1)^{m-2} T_0^m A_0 \right. \\
&\quad \left. + \binom{n+1}{m+1} \mathcal{C}_{anom} A_0^{m+1} \right] \\
J_m + q U_m &= \gamma \left[ (n+1-m) C_{m-1} (-1)^{m-2} T_0^m - (n+1) \binom{n}{m} \mathcal{C}_{anom} A_0^m \right] \\
S_m + s U_m &= \gamma \left[ -(m+1) C_m (-1)^{m-1} T_0^m + C_0 (-1)^{0-1} \binom{n}{m} A_0^m \right]
\end{aligned} \tag{B.10}$$

which matches with expressions from the partition function.

The corresponding consistent currents can be obtained via the relations

$$\begin{aligned}
\bar{\mathcal{G}}_{anom}^{Cov} &= \bar{\mathcal{G}}_{anom}^{Consistent} - \mu n \mathcal{C}_{anom} \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} \\
\bar{J}_{anom}^{Cov} &= \bar{J}_{anom}^{Consistent} + n \mathcal{C}_{anom} \hat{\mathcal{A}} \wedge \mathcal{F}^{n-1} \\
\bar{J}_{S,anom}^{Cov} &= \bar{J}_{S,anom}^{Consistent} \\
\bar{q}_{anom}^{Cov} &= \bar{q}_{anom}^{Consistent}
\end{aligned} \tag{B.11}$$

In particular we have

$$\begin{aligned}
& -\frac{1}{T} \bar{\mathcal{G}}_{anom}^{Consistent} \\
& = \frac{1}{T_0} \sum_{m=1}^n \left[ C_m (-1)^{m-1} T_0^{m+1} - C_0 (-1)^{0-1} \binom{n}{m} T_0 A_0^m \right. \\
& \quad \left. - \binom{n}{m+1} \mathcal{C}_{anom} A_0^{m+1} \right] (da)^{m-1} (dA)^{n-m} \wedge (dt + a) \\
& \quad - \frac{1}{T_0} [n \mathcal{C}_{anom} A_0 + C_0 T_0] A \wedge (dA + A_0 da)^{n-1} \\
& \quad - \frac{(n-1)}{T_0} [n \mathcal{C}_{anom} A_0 + C_0 T_0] A \wedge dA_0 \wedge (dt + a) \wedge (dA + A_0 da)^{n-2}
\end{aligned} \tag{B.12}$$

## C. Variational formulae in forms

The energy current is defined via the relation

$$\begin{aligned}
q_\mu dx^\mu & \equiv -T_{\mu\nu} u^\mu dx^\nu \\
& = -\gamma T_{00} (dt + a) - \gamma g_{ij} T_0^i dx^j
\end{aligned} \tag{C.1}$$

Hence its Hodge dual is (See D for the definition of Hodge dual)

$$\bar{q} = \gamma^3 T_{00} d\mathcal{V}_{d-1} + \gamma T_0^i (dt + a) \wedge (d\Sigma_{d-2})_i \tag{C.2}$$

We take the following relations<sup>13</sup> from Eqn(2.16) of [16]

$$\begin{aligned}
\gamma T_{00} d\mathcal{V}_{d-1} & = \frac{\delta}{\delta\gamma} (T_0 \ln \mathcal{Z}) \\
T_0^i d\mathcal{V}_{d-1} & = dx^i \wedge T_0^j (d\Sigma_{d-2})_j = \left[ \frac{\delta}{\delta a_i} - A_0 \frac{\delta}{\delta A_i} \right] (T_0 \ln \mathcal{Z})
\end{aligned} \tag{C.3}$$

where the independent variables are  $\{\gamma, a, g^{ij}, A_0, A, T_0, \mu_0\}$ . Converting into forms

$$\begin{aligned}
\bar{q} & = \left[ \gamma^2 \frac{\delta}{\delta\gamma} + \gamma (dt + a) \wedge \frac{\delta}{\delta a} - \gamma A_0 (dt + a) \wedge \frac{\delta}{\delta A} \right] (T_0 \ln \mathcal{Z}) \\
& = \left[ \gamma^2 \frac{\delta}{\delta\gamma} + \gamma (dt + a) \wedge \frac{\delta}{\delta a} - \mu (dt + a) \wedge \frac{\delta}{\delta A} \right] (T_0 \ln \mathcal{Z})
\end{aligned} \tag{C.4}$$

Similarly for the charge current

$$\begin{aligned}
-\gamma^2 J_0 d\mathcal{V}_{d-1} & = \frac{\delta}{\delta A_0} (T_0 \ln \mathcal{Z}) \\
J^i d\mathcal{V}_{d-1} & = dx^i \wedge J^j (d\Sigma_{d-2})_j = \frac{\delta}{\delta A_i} (T_0 \ln \mathcal{Z})
\end{aligned} \tag{C.5}$$

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<sup>13</sup>we remind the reader that  $\gamma \equiv e^{-\sigma}$  and  $d\mathcal{V}_{d-1} = d^{d-1}x \sqrt{-\det g_d}$

which implies

$$\begin{aligned}\bar{J} &\equiv -\gamma^2 J_0 d\mathcal{V}_{d-1} - J^i(dt + a) \wedge (d\Sigma_{d-2})_i \\ &= \left[ \frac{\delta}{\delta A_0} - (dt + a) \wedge \frac{\delta}{\delta A} \right] (T_0 \ln \mathcal{Z})\end{aligned}\tag{C.6}$$

Putting  $T_0 \ln \mathcal{Z} = -\gamma^{-1} \bar{\mathcal{G}}$  we can write

$$\begin{aligned}\bar{J} &\equiv -\frac{\partial \bar{\mathcal{G}}}{\partial \mu} = -\gamma^{-1} \left[ \frac{\delta}{\delta A_0} - (dt + a) \wedge \frac{\delta}{\delta A} \right] \bar{\mathcal{G}} \\ \bar{J}_S &\equiv -\frac{\partial \bar{\mathcal{G}}}{\partial T} = -\gamma^{-1} \frac{1}{T_0} \left[ \gamma \frac{\delta}{\delta \gamma} + (dt + a) \wedge \frac{\delta}{\delta a} - A_0 \frac{\delta}{\delta A_0} \right] \bar{\mathcal{G}} \\ \bar{q} &= \bar{\mathcal{G}} + T \bar{J}_S + \mu \bar{J}\end{aligned}\tag{C.7}$$

## D. Convention for Forms

The inner product between two 1-forms  $J \equiv J_0(dt + a) + g_{ij}J^i dx^j$  and  $J' \equiv J'_0(dt + a) + g_{ij}(J')^i dx^j$  is given in terms of the KK-invariant components as

$$\langle J, J' \rangle \equiv -\gamma^2 J_0 J'_0 + g_{ij} J^i (J')^j\tag{D.1}$$

In general, the exterior derivative of a p-form

$$A_p \equiv \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

is given by

$$\begin{aligned}(dA)_{p+1} &\equiv \frac{1}{p!} \partial_\lambda A_{\mu_1 \dots \mu_p} dx^\lambda \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \\ &= \frac{1}{(p+1)!} [\partial_{\mu_1} A_{\mu_2 \dots \mu_{p+1}} + \text{cyclic}] dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}}\end{aligned}\tag{D.2}$$

The Levi-Civita tensor  $\varepsilon^{\mu_1 \dots \mu_d}$  is defined as the completely antisymmetric tensor with

$$\varepsilon^{012 \dots (d-1)} = \frac{1}{\sqrt{-\det g_d}} = \frac{1}{\gamma^{-1} \sqrt{\det g_{d-1}}}$$

We will also often define the spatial Levi-Civita tensor  $\epsilon^{i_1 i_2 \dots i_{d-1}}$  such that

$$\epsilon^{12 \dots (d-1)} = \frac{1}{\sqrt{\det g_{d-1}}}$$

which is related to its spacetime counterpart via

$$\epsilon^{i_1 i_2 \dots i_{d-1}} = \gamma^{-1} \varepsilon^{0 i_1 i_2 \dots i_{d-1}}$$



Let us define the spatial volume  $(d-1)$ -form as

$$\begin{aligned}
d\mathbb{V}_{d-1} &\equiv \gamma^{-1} \epsilon_{i_1 \dots i_{d-1}} dx^{i_1} \otimes \dots \otimes dx^{i_{d-1}} \\
&= \frac{1}{(d-1)!} \gamma^{-1} \epsilon_{i_1 \dots i_{d-1}} dx^{i_1} \wedge \dots \wedge dx^{i_{d-1}} \\
&= d^{d-1} x \gamma^{-1} \sqrt{\det g_{d-1}} \\
&= d^{d-1} x \sqrt{-\det g_d}
\end{aligned} \tag{D.3}$$

where  $\epsilon_{i_1 \dots i_{d-1}}$  is the spatial Levi-Civita symbol. The form  $d\mathbb{V}_{d-1}$  transforms like a vector with a lower time-index and hence is KK-invariant.

Define the spatial area  $(d-2)$ -form as

$$\begin{aligned}
(d\Sigma_{d-2})_j &\equiv \gamma^{-1} \epsilon_{ji_1 \dots i_{d-2}} dx^{i_1} \otimes \dots \otimes dx^{i_{d-2}} \\
&= \frac{1}{(d-2)!} \gamma^{-1} \epsilon_{ji_1 \dots i_{d-2}} dx^{i_1} \wedge \dots \wedge dx^{i_{d-2}}
\end{aligned} \tag{D.4}$$

This transforms like a vector with a lower time-index and a lower spatial index but is antisymmetric in these two indices and is hence KK-invariant. The area  $(d-2)$ -form satisfies

$$dx^i \wedge (d\Sigma_{d-2})_j = d\mathbb{V}_{d-1} \delta_j^i$$

The Hodge-dual of a 1-form  $J \equiv J_0(dt + a) + g_{ij}J^i dx^j$  is defined as

$$\bar{J} = -\gamma^2 J_0 d\mathbb{V}_{d-1} - J^i (dt + a) \wedge (d\Sigma_{d-2})_i \tag{D.5}$$

This is defined such that

$$J' \wedge \bar{J} = \langle J', J \rangle (dt + a) \wedge d\mathbb{V}_{d-1} = \langle J', J \rangle dt \wedge d\mathbb{V}_{d-1} \tag{D.6}$$

In particular

$$d\bar{J} = (\nabla_\mu J^\mu) dt \wedge d\mathbb{V}_{d-1} \tag{D.7}$$

One often useful formula is this

$$\begin{aligned}
\bar{J} &= \hat{\mathcal{A}} \wedge (d\hat{\mathcal{A}})^{n-1} \\
&\text{is equivalent to} \\
J^\mu &= \left[ \varepsilon \hat{\mathcal{A}} (\partial \hat{\mathcal{A}})^{n-1} \right]^\mu
\end{aligned} \tag{D.8}$$

Let us take another example which will recur throughout our paper - say we are given that the Hodge-dual of a 1-form  $J \equiv J_0(dt + a) + g_{ij}J^i dx^j$  is

$$-\bar{J} = A \wedge (da)^{m-1} (dA)^{n-m} + A_0 (dt + a) \wedge (da)^{m-1} (dA)^{n-m}$$

where  $a = a_i dx^i$  and  $A = A_i dx^i$  are two arbitrary 1-forms with only spatial components.

Then we can invert the Hodge-dual using the following statement

$$\begin{aligned}\bar{J} &= -A \wedge (da)^{m-1} (dA)^{n-m} - A_0 (dt + a) \wedge (da)^{m-1} (dA)^{n-m} \\ &\text{is equivalent to} \\ J_0 &= \gamma^{-1} [\epsilon A (da)^{m-1} (dA)^{n-m}] \\ J^i &= \gamma A_0 [\epsilon (da)^{m-1} (dA)^{n-m}]^i\end{aligned}\tag{D.9}$$

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